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REVISITING THE PENTA-DIMENSIONAL DEFINITION OF MASS AS A MOMENTUM

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ABSTRACT

Mass is hypothesized to be a momentum in a 5D Kaluza-Klein setting. The resulting Maxwellian field is not interpreted as electromagnetism but, along with the dilaton, as a fifth force. Despite their failure to explain the observation of dark matter filaments, the possibility that either the Maxwellian field either the dilaton field is responsible for the rotational curve of galaxies is discussed. Similarities between the dilaton field and the Higgs field are discussed, not only at a scalar level but at a vectorial level. The electroweak gauge bundle is seen as a subbundle of the 5D tangent frame bundle. The invisibility and the global topology of the fifth dimension is discussed. An expanding cosmic wavefront theory is suggested, then discarded. Assuming that mass is a momentum, it is shown that positrons should have a negative mass and that there should be roughly no gravitational attraction between neighbouring galaxies.

Contents

1. INTRODUCTION

The 4D Klein-Gordon equation and the 5D wave equation are respectively:

$$0 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m^2c^2}{\hbar^2}\right)\psi \quad (1)$$

$$0 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial s^2}\right)\psi \quad (2)$$

Because they look a lot alike, let's hypothesize that *they are the same equation.* Thus, ψ is an eigenfunction of a *mass operator*:

$$\hat{m}c = -i\hbar\frac{\partial}{\partial s} \quad (3)$$

This amounts to interpret mass no more as the 4D $(+,-,-,-)$ Minkowskian norm of a timelike 4-momentum:

$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = (mc)^2$$

but as a fifth component $mc = p_4$ of a lightlike 5-momentum:

$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - (p_4)^2 = 0$$

Let's call this the *mass = momentum hypothesis* (MMH). Such a penta-dimensional perspective was discussed in (Aubin-Cadot 2018).

The geometrical setup of the MMH is a 5D Kaluza-Klein (KK) context (Kaluza 1921), (Klein 1926). But, the interpretation of the geometrical constituents of KK theory is different than conventional KK theory. The momentum in the fifth dimension is interpreted as mass, not as an electrical charge. Thus, the MMH mass is not a second hand effective mass coming out of an electric charge. The MMH does not directly involve electromagnetism. The resulting *KK Maxwellian field* is interpreted as being part of a fifth force, not as conventional electromagnetism. This fifth force is suggested to explain the rotational curve of galaxies. However, it is incompatible with the observation of dark matter filaments because it is shown that the fifth force does not bend 4D light rays.

At first, the MMH is not a unification theory of electromagnetism with gravity because it is not motivated by such a unification, only by a geometrical definition of mass. Electromagnetism is part of a broader electroweak $\mathrm{U}(1)\times\mathrm{SU}(2)$ gauge theory involving a *Higgs field*. When digging in the analogies between the *KK dilaton field* and the Higgs field, the electroweak structural group seems to be part of the bigger $\mathrm{O}(1,4)$ structural group resulting from $(+,-,-,-,-)$ geometry. However, despite the analogies, the MMH mass and the mechanism by which the Higgs gives mass are different.

Because in the MMH context the electroweak gauge theory lives *over* a 5D space, the MMH could be confused with the so-called *Universal Extra Dimension* (UED) theory (Arkani-Hamed et al. 1998), (Dienes & al. 1998), (Appelquist et al. 2001). Contrary to the UED theory, in the MMH theory massive particles are not *free to move* in the fifth dimension as if is was just another usual dimension. This is because the 4D timelike geodesics of massive particles correspond to 5D lightlike geodesics.

According to the MMH theory, because photons are massless, their momentum in the fifth dimension vanishes. Thus, we do not see the fifth dimension. Because of momentum conservation and because photons are massless, it follows that positrons have a negative mass. It is shown that such a negative mass does not contradict the expected Lorentz dynamic of a point particle positron.

Because of (Bonnor 1969), when identifying 4D timelike dust with 5D lightlike dust, gravity seems to be turned off between neighbouring galaxies. Nevertheless, even if gravity is turned off between neighbouring galaxies, they do bend 4D light rays as expected.

<u>*Outline:*</u>

- §2-5 are a pot-pourri of standard definitions, formulas, notation and sign conventions regarding Riemannian geometry, general relativity and G-principal bundle theory.
- §6 is a smooth transition from the theory of principal bundles to $(\mathbb{R},+)$ KK theory.
- §7 contains the main KK decomposition formulas for the Christoffel symbols, the Ricci curvature and the scalar curvature. I follow (Williams 2015), but with a $(+,-,-,-,-)$ sign convention instead of a $(+,-,-,-,+)$. The decomposition of the 5D Hilbert-Einstein action integral to a 4D one is also given and the field equations of the metric, of the Maxwellian field and of the dilaton field are given.
- §8 is devoted to go from the 5D wave equation to the 4D equations of motion of point particles.
- §9 recalls the electromagnetic interpretation of KK theory and its implication that particles are $\approx 10^{20}$ too heavy.
- §10 gives a precise mathematical formulation of the above mentioned MMH and the definition of a mass operator is sketched.
- §11 discusses the possibility that either the Maxwellian field either the dilaton field could be taken responsible for the rotational curves of galaxies.
- §12 suggests a way to get rid of the dilaton field. Although such a direction is not taken, the same mathematical considerations will serve to relate the dilaton to the Higgs.
- §13 suggests a way to define a mass for the dilaton field in analogy with how the mass of the Higgs is defined. However, this defined mass is not compatible with the MMH.
- §14 relates the dilaton and the Higgs not only at a scalar level but at a vectorial level. The electroweak gauge bundle is seen as a subbundle of the 5D tangent frame bundle.
- §15 shows that, according to MMH, positrons should a negative mass. The idea of a mass operator is discussed once more. The invisibility of the fifth dimension is discussed. A expanding cosmic wavefront theory is discarded. It is argued that the fifth dimension should be a circle S^1, not $\mathbb{R}$. It is also argued that the MMH implies an absence of gravity between neighbouring galaxies.

2. RIEMANNIAN GEOMETRY

Let's recall some standard definitions of Riemannian geometry. Let (Q,g) be an n-dimensional pseudo-Riemannian manifold. Consider local coordinates (x^i) on $U \subset Q$. Let $\partial_i := \partial/\partial x^i$ be defined as $dx^i(\partial_j) = \delta^i_j$ where δ^i_j equals 1 if $i=j$ and 0 if $i \neq j$. The metric g is locally written:

$$g|_U = g_{ij} dx^i \otimes dx^j \tag{4}$$

where $g_{ij} := g(\partial_i, \partial_j)$. Repeated indices are summed over their range. Let g^{ij} be the coefficients of the inverse matrix $[g^{ij}] := [g_{ij}]^{-1}$, i.e. $\delta^i_j = g^{ik} g_{kj}$. Following (Landau & Lifschitz 1971), the *Christoffel symbols* (86.3), the *Riemann curvature tensor* (91.4), the *Ricci curvature tensor* (92.9), the *scalar curvature* (92.12) and the *Einstein tensor* (95.5) of the metric g are respectively given by:

$$\Gamma^k_{ij} := (1/2) g^{km} (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}) \tag{5}$$
$$R^l_{kij} := \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik} \tag{6}$$
$$R_{ij} := R^k_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl} \tag{7}$$
$$R := g^{ij} R_{ij} \tag{8}$$
$$G_{ij} := R_{ij} - (1/2) g_{ij} R \tag{9}$$

The *Riemannian musicality isomorphisms* are:

$$\begin{aligned} \flat &: TQ \to T^*Q \\ v = v^i\partial_i &\mapsto v^\flat := g(v,\cdot) = g_{ij} v^i dx^j \end{aligned}$$

$$\begin{aligned} \sharp := \flat^{-1} &: T^*Q \to TQ \\ \alpha = \alpha_i dx^i &\mapsto \alpha^\sharp := g^{ij}\alpha_i \partial_j \end{aligned}$$

Let:

$$\begin{aligned} \|v\|^2_g &:= g(v,v) = g_{ij} v^i v^j \\ \|\alpha\|^2_g &:= g(\alpha^\sharp, \alpha^\sharp) = g^{ij}\alpha_i\alpha_j \end{aligned}$$

When a vector v and a 1-form α are mutually musical, we have $\|v\|_g = \|\alpha\|_g$. Let:

$$\begin{aligned} \det[g] &:= \det[g_{ij}] \\ |g| &:= |\det[g]| \\ d^n x &:= dx^1 \wedge .. \wedge dx^n \\ \Omega_g &:= |g|^{1/2} d^n x \end{aligned}$$

The *Levi-Civita covariant derivative* $\nabla_i := \nabla_{\partial_i}$ of tensors on Q is given in local coordinates by the Leibniz product rule and by:

$$\nabla_i f := \partial_i f \qquad \nabla_i(\partial_j) := \Gamma^k_{ij}\partial_k \qquad \nabla_i(dx^k) := -\Gamma^k_{ij} dx^j$$

On functions, one should not confuse the *gradient vector field of a function*:

$$\begin{aligned} \nabla f &:= (df)^\sharp \\ &= g^{ij}(\partial^i f)\partial_j \end{aligned}$$

and the *covariant derivative of a function*:

$$\begin{aligned} \nabla f &= df \\ &= (\partial_i f) dx^i \end{aligned}$$

These two ∇ are not the same. For this reason, on tensors ∇ will denote a covariant derivative but on functions ∇ will denote the gradient. Also, as an abuse of notation, the covariant derivative ∇T of a tensor T will often be denoted $\nabla T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$. For example:

$$\nabla_i F_{jk} = \partial_i F_{jk} - \Gamma^l_{ij} F_{lk} - \Gamma^l_{ik} F_{jl}$$

The *covariant Hessian* of a function is:

$$\begin{aligned} H_{ij}(f) &:= (\nabla df)(\partial_i, \partial_j) \\ &= \left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right) f \end{aligned}$$

The *covariant divergence* of a vector field $v = v^i\partial_i$ is defined implicitly as:

$$\mathrm{Div}(v) \cdot \Omega_g = \mathcal{L}_v \Omega_g$$

The covariant divergence is given explicitly as:

$$\begin{aligned} \mathrm{Div}(v) &= \partial_i v^i + v^i \Gamma^k_{ik} \\ &= |g|^{-1/2}\partial_i \left(|g|^{1/2} v^i\right) \end{aligned}$$

In particular, we have:

$$\mathrm{Div}(\partial_i) = \Gamma^k_{ik} = |g|^{-1/2}\partial_i\left(|g|^{1/2}\right)$$

For α a differential 1-form we have:

$$\mathrm{Div}(\alpha^\sharp) = g^{ij}(\partial_i \alpha_j - \Gamma^k_{ij}\alpha_k)$$

The *Laplace-Beltrami operator* acting on functions is:

$$\begin{aligned} \Delta f &:= \mathrm{Div}(\nabla f) \\ &= g^{ij}\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right) f \\ &= g^{ij} H_{ij}(f) \end{aligned}$$

On functions, the Laplace-Beltrami operator equals minus the Laplace-de Rham operator, i.e. $\Delta_{\mathrm{LB}} f = -\Delta_{\mathrm{LdR}} f$ where $\Delta_{\mathrm{LdR}} f := \delta df$ where δ is the Hodge codifferential. Here Δ will always denote Δ_{LB}.

In a $(+,-,-,-)$ pseudo-Riemannian setting, Δ is denoted by the *d'Alembertian operator* $\Box$. In a

$(+,-,-,-,-)$ pseudo-Riemannian setting, Δ is denoted by the *Souriau operator* $\pentagon$.

Let $\gamma(\lambda)$ be a parametrized curve in (Q,g) and denote by a dot the derivative $d/d\lambda$. γ is said to be a *geodesic* of (Q,g) if it satisfies the *geodesic equation* $\ddot{\gamma}^k+\Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j=0$. γ is said to be an *integral curve* of a vector field v if it satisfies the *integral curve equation* $\dot{\gamma}=v\circ\gamma$. γ is said to be a *gradient curve* of a function f if it is an integral curve of the gradient vector field ∇f, i.e. $\dot{\gamma}=(\nabla f)\circ\gamma$. A vector field $v\in\mathfrak{X}(Q)$ is a *Killing vector field* of g if $\mathcal{L}_v g=0$.

Under a *conformal rescaling* $\hat{g}=e^{2\varphi}g$ we have:

$$
\begin{aligned}
\hat{g}_{ij} &= e^{2\varphi}g_{ij}\\
\hat{g}^{ij} &= e^{-2\varphi}g^{ij}\\
\hat{\Gamma}^k_{ij} &= \Gamma^k_{ij}+\delta^k_i\partial_j\varphi+\delta^k_j\partial_i\varphi-g_{ij}\nabla^k\varphi\\
\hat{R}_{ij} &= R_{ij}-(n-2)\left(H_{ij}(\varphi)-(\partial_i\varphi)(\partial_j\varphi)\right)\\
&\quad -\left(\Delta\varphi+(n-2)\|d\varphi\|^2_g\right)g_{ij}\\
\hat{R} &= e^{-2\varphi}[R-2(n-1)\Delta\varphi\\
&\quad -(n-2)(n-1)\|d\varphi\|^2_g]\\
|\hat{g}|^{1/2} &= e^{n\varphi}|g|^{1/2}\\
\Omega_{\hat{g}} &= e^{n\varphi}\Omega_g\\
\hat{R}\Omega_{\hat{g}} &= e^{(n-2)\varphi}[R-2(n-1)\Delta\varphi\\
&\quad -(n-2)(n-1)\|d\varphi\|^2_g]\Omega_g\\
\alpha^{\hat{\sharp}} &= e^{-2\varphi}\alpha^\sharp\\
v^{\hat{\flat}} &= e^{2\varphi}v^\flat\\
\|v\|^2_{\hat{g}} &= e^{2\varphi}\|v\|^2_g\\
\|\alpha\|^2_{\hat{g}} &= e^{-2\varphi}\|\alpha\|^2_g\\
\widehat{\mathrm{Div}}(v) &= \mathrm{Div}(v)+nd\varphi(v)\\
\widehat{\mathrm{Div}}(\alpha^{\hat{\sharp}}) &= e^{-2\varphi}(\mathrm{Div}(\alpha^\sharp)+(n-2)\alpha(\nabla\varphi))\\
\hat{\nabla}f &= e^{-2\varphi}\nabla f\\
\hat{\Delta}f &= e^{-2\varphi}(\Delta f+(n-2)(df)(\nabla\varphi))
\end{aligned}
$$

3. GENERAL RELATIVITY

Let's recall some standard definitions of general relativity (Landau & Lifschitz 1971). Let (Q,g) be a 4-dimensional $(+,-,-,-)$ pseudo-Riemannian manifold called *spacetime*. The *Einstein constant* is defined as $\kappa:=8\pi G/c^4$. Let L be some Lagrangian density. The *Hilbert energy-momentum tensor* of L is defined as (94.4):

$$
\begin{aligned}
T_{ij} &:= 2\frac{1}{|g|^{1/2}}\frac{\partial}{\partial g^{ij}}(L|g|^{1/2})\\
&= 2\frac{\partial L}{\partial g^{ij}}-g_{ij}L
\end{aligned}
$$

The *Laue scalar* of L is defined as:

$$T:=g^{ij}T_{ij} \tag{10}$$

The *Hilbert-Einstein (HE) action integral* S_{HE} is defined as:

$$S_{\mathrm{HE}}[g,S]:=\int_Q\left(\frac{1}{2\kappa}R+L\right)\Omega_g \tag{11}$$

Using the identity:

$$
\begin{aligned}
G_{ij} &= \frac{1}{|g|^{1/2}}\frac{\delta}{\delta g^{ij}}(R|g|^{1/2})\\
&= R_{ij}-\frac{1}{2}g_{ij}R
\end{aligned}
$$

the Euler-Lagrange (EL) equation according to variations in g_{ij} of the HE action integral gives the *Einstein field equation (EFE) of general relativity* (95.5):

$$G_{ij}=\kappa T_{ij} \tag{12}$$

Tracing both sides of the Einstein equation (12) with g^{ij} and using the fact that $g^{ij}G_{ij}=-R$, one finds (95.7):

$$R=-\kappa T \tag{13}$$

Let's define this tensor:

$$K_{ij}:=T_{ij}-\frac{1}{2}g_{ij}T \tag{14}$$

Using (13), the EFE (12) is equivalent to the *trace-reversed EFE* (95.8):

$$R_{ij}=\kappa K_{ij} \tag{15}$$

The *electromagnetic (EM) Lagrangian density*, the *EM energy-momentum tensor* (94.8), the *EM Laue scalar* and the *EM K_{ij} tensor* are respectively:

$$L=-\frac{1}{4\mu_0}\|F\|^2_g=-\frac{1}{4\mu_0}g^{ik}g^{jl}F_{ij}F_{kl} \tag{16}$$

$$T_{ij}=-\frac{1}{\mu_0}\left(g^{kl}F_{ik}F_{jl}-\frac{1}{4}g_{ij}\|F\|^2_g\right) \tag{17}$$

$$T=0 \tag{18}$$

$$K_{ij}=T_{ij} \tag{19}$$

For a real constant a and a Lagrangian density defined over a real function f as follow, we have:

$$
\begin{aligned}
L &= \frac{a}{2}\|df\|^2_g=\frac{a}{2}g^{ij}(\partial_i f)(\partial_j f)\\
T_{ij} &= a\left((\partial_i f)(\partial_j f)-\frac{1}{2}g_{ij}\|df\|^2_g\right)\\
T &= -a\|df\|^2_g\\
K_{ij} &= a(\partial_i f)(\partial_j f)
\end{aligned}
$$

For this last L, the EL equation corresponding to variations in f of (11) gives this other field equations:

$$\partial_i\left(\frac{\partial L}{\partial(\partial_i f)}|g|^{1/2}\right)=\frac{\partial L}{\partial f}|g|^{1/2} \tag{20}$$

which is here $\Box f = 0$. For a *perfect fluid* of *pressure* p and *energy density* ρ_e we have (94.9):

$$
\begin{aligned}
T_{ij} &= (\rho_e + p)u_i u_j / c^2 - p g_{ij} \\
T &= \rho_e - 3p \\
K_{ij} &= (\rho_e + p)u_i u_j / c^2 - \frac{1}{2} g_{ij}(\rho_e - p)
\end{aligned}
$$

For a *timelike dust* we have $p = 0$ and $\|u\|_g^2 = c^2$. For a *lightlike dust* we have $p = 0$ and $\|u\|_g^2 = 0$. For an *isotropic radiation fluid* we have $p = \rho_3/3$ and $\|u\|_g^2 = c^2$.

4. G-PRINCIPAL BUNDLE THEORY

Now, let's recall some standard definitions regarding Lie groups and G-principal bundle theory (Kobayashi & Nomizu 1963). Let G be a smooth finite-dimensional *Lie group*. Let $e \in G$ denote its *identity element*. Let $\mathfrak{g} := \mathrm{Lie}(G) := T_e G$ denote its *Lie algebra*. Let:

$$
\begin{aligned}
L &: G \to \mathrm{Aut}(G);\ \lambda \mapsto L_\lambda \\
R &: G \to \mathrm{Aut}(G);\ \lambda \mapsto R_\lambda \\
\iota &: G \to \mathrm{Aut}(G);\ \lambda \mapsto \iota_\lambda
\end{aligned}
$$

denote respectively the *left* group action, the *right* group action and the *interior product*:

$$
\begin{aligned}
L_{\lambda_1}(\lambda_2) &= \lambda_1 \lambda_2 \\
R_{\lambda_1}(\lambda_2) &= \lambda_2 \lambda_1 \\
\iota_{\lambda_1}(\lambda_2) &= \lambda_1 \lambda_2 \lambda_1^{-1}
\end{aligned}
$$

The *adjoint representation* is defined as:

$$
\begin{aligned}
\mathrm{Ad} &: G \to \mathrm{Aut}(\mathfrak{g}) \\
&\lambda \mapsto \mathrm{Ad}_\lambda := ((\iota_\lambda)_*|_e : \mathfrak{g} \to \mathfrak{g})
\end{aligned}
$$

A direct calculation shows that the two maps:

$$
\begin{aligned}
L_*|_e &: \mathfrak{g} \to \mathfrak{X}(G) \\
R_*|_e &: \mathfrak{g} \to \mathfrak{X}(G)
\end{aligned}
$$

are explicitly given on each $\xi \in \mathfrak{g}$ and at each $\lambda \in G$ by:

$$
\begin{aligned}
(L_*|_e(\xi))|_\lambda &= (R_\lambda)_*(\xi) \\
(R_*|_e(\xi))|_\lambda &= (L_\lambda)_*(\xi)
\end{aligned}
$$

A $\xi \in \mathfrak{g}$ induces a *left-invariant* vector field on G:

$$\xi^\circ := R_*|_e(\xi) \in \mathfrak{X}(G)$$

The *Maurer-Cartan* differential 1-form $\theta \in \Omega^1(G;\mathfrak{g})$ on G is pointwise defined at each $\lambda \in G$ as:

$$\theta|_\lambda := (L_{\lambda^{-1}})_* : T_\lambda G \to T_e G$$

The Maurer-Cartan 1-form is constant on left-invariant vector fields, i.e. for all $\xi \in \mathfrak{g}$ we have:

$$\theta(\xi^\circ) = \xi \tag{21}$$

For all $g \in G$, the Maurer-Cartan 1-form satisfies:

$$(L_\lambda)^*\theta = \theta \tag{22}$$

$$(R_\lambda)^*\theta = \mathrm{Ad}_\lambda^{-1} \circ \theta \tag{23}$$

The first equation is the *left-invariance* of the Maurer-Cartan 1-form while the second is its *right-equivariance*.

Now, let G be a Lie group, call it the *structural group*. A *right G-principal bundle* P over X is a *fiber bundle* $\pi : P \to X$ whose fibers are the orbits of a smooth free right group action:

$$
\begin{aligned}
\Phi &: G \to \mathrm{Diff}(P) \\
&\lambda \mapsto \Phi_\lambda
\end{aligned}
$$

where the *canonical projection* π is differentiable and where the *total space* P is locally trivial. The right group action Φ is also denoted $\Phi_\lambda(a) = a \cdot \lambda$ for $a \in P$ and $\lambda \in G$. The "right" aspect of Φ means that $\Phi : G \to \mathrm{Diff}(P)$ is an antihomomorphism. The local triviality of P means that each $x \in X$ admits a neighbourhood U_μ and a G-equivariant map $\phi_\mu : \pi^{-1}(U) \to G$, i.e. $\phi_\mu(a \cdot \lambda) = (\phi_\mu(a))\lambda$, $\forall a \in P$, $\forall \lambda \in G$, such that the map:

$$\Psi_\mu : \pi^{-1}(U_\mu) \to U_\mu \times G;\ \ a \mapsto (\pi(a), \phi_\mu(a))$$

is a diffeomorphism. To a *local trivialization* Ψ_μ corresponds a unique *local trivializing section*:

$$s_\mu : U_\mu \to \pi^{-1}(U_\mu)$$

defined over U_μ by:

$$\phi_\mu \circ s_\mu = e \in G$$

To a local trivializing section s_μ over U_μ corresponds a family of local trivializing sections:

$$s_{\mu,\lambda} := s_\mu \cdot \lambda$$

over U_μ so that $s_\mu = s_{\mu,e}$. Denote the *graph* of $s_{\mu,\lambda}$ by $\Gamma_{\mu,\lambda} := s_{\mu,\lambda}(U_\mu) \subset \pi^{-1}(U_\mu)$. For a given s_μ, the set of graphs $\{\Gamma_{\mu,\lambda} : \lambda \in G\}$ foliate $\pi^{-1}(U_\mu)$. The inverse map of a local trivialization Ψ_μ is given by:

$$\Psi_\mu^{-1} : U_\mu \times G \to \pi^{-1}(U_\mu);\ \ (x, \lambda) \mapsto s_\mu(x) \cdot \lambda = s_{\mu,\lambda}(x)$$

When $U_{\mu\nu} := U_\mu \cap U_\nu \neq \emptyset$, two local trivializations:

$$
\begin{aligned}
\Psi_\mu &: \pi^{-1}(U_\mu) \to U_\mu \times G;\ \ a \mapsto (\pi(a), \phi_\mu(a)) \\
\Psi_\nu &: \pi^{-1}(U_\nu) \to U_\nu \times G;\ \ a \mapsto (\pi(a), \phi_\nu(a))
\end{aligned}
$$

induce a *transition function*:

$$\Psi_{\mu\nu} : U_{\mu\nu} \to G$$

defined for all $a \in \pi^{-1}(U_\mu \cap U_\nu)$ by:

$$\Psi_{\mu\nu}(\pi(a)) := \widetilde{\Psi}_{\mu\nu}(a) := \phi_\mu(a)\phi_\nu(a)^{-1}$$

Because $\widetilde{\Psi}_{\mu\nu}$ is G-invariant, $\Psi_{\mu\nu}$ is well defined. It can be shown that:

$$\begin{aligned} \Psi_{\mu\nu}\Psi_{\nu\gamma} &= \Psi_{\mu\gamma} \\ \Psi_{\mu\nu}(x) &= \Psi_{\nu\mu}(x)^{-1}, \quad \forall x \in U_{\mu\nu} \\ s_\nu &= s_\mu \cdot \Psi_{\mu\nu} \\ \pi \circ \Psi_\mu^{-1}(x,\lambda) &= x, \quad \forall (x,\lambda) \in U_\mu \times G \\ \phi_\mu \circ \Psi_\mu^{-1}(x,\lambda) &= \lambda, \quad \forall (x,\lambda) \in U_\mu \times G \end{aligned}$$

Denote by:

$$\xi^\bullet := \Phi_*|_e(\xi) \in \mathfrak{X}(P)$$

the *fundamental vector field* on P corresponding to $\xi \in \mathfrak{g}$. For all $a \in P$, for all $\lambda \in G$ and for all $\xi \in \mathfrak{g}$, fundamental vector fields satisfy:

$$(\Phi_\lambda)_*(\xi^\bullet|_a) = (\mathrm{Ad}_\lambda^{-1} \circ \xi)^\bullet|_{a\cdot\lambda}$$

A *connection 1-form* A over P is a $\mathfrak{g}$-valued differential 1-form $A \in \Omega^1(P;\mathfrak{g})$ that satisfies:

$$A(\xi^\bullet) = \xi, \quad \forall \xi \in \mathfrak{g} \tag{24}$$
$$(\Phi_\lambda)^* A = \mathrm{Ad}_\lambda^{-1} \circ A, \quad \forall \lambda \in G \tag{25}$$

The infinitesimal equivalent version of (25) is:

$$\mathcal{L}_{\xi^\bullet} A = -[\xi, A], \quad \forall \xi \in \mathfrak{g} \tag{26}$$

On P there is a canonical *vertical distribution* $V \subset TP$ pointwise spanned at each point $a \in P$ by all fundamental vector fields:

$$V_a := \mathbb{R}\langle \xi^\bullet|_a : \xi \in \mathfrak{g}\rangle \subset T_aP \tag{27}$$

A connection 1-form A on P defines a *horizontal distribution* $H \subset TP$ pointwise defined as being the kernel of the connection form:

$$H_a := \ker(A_a) \subset T_aP \tag{28}$$

The vertical distribution V is G-invariant and integrable as a foliation whose leaves are the fibers of P. The horizontal distribution H is also G-invariant because A is Ad-equivariant, but it is not always integrable as a foliation. The vertical distribution V and a horizontal distribution H together satisfy:

$$H + V = TP \tag{29}$$
$$H \cap V = \{0\} \tag{30}$$

To this splitting of TP corresponds respectively a *vertical projection* and a *horizontal projection*:

$$\mathrm{ver} : TP \to V \tag{31}$$
$$\mathrm{hor} : TP \to H \tag{32}$$

explicitly given at each $a \in P$ and on each $v \in T_aP$ by:

$$\mathrm{ver}|_a(v) = (A(v))^\bullet|_a \tag{33}$$
$$\mathrm{hor}|_a(v) = v - \mathrm{ver}|_a(v) \tag{34}$$

so that $v = \mathrm{hor}(v) + \mathrm{ver}(v)$. When a trivialization Ψ_μ is given, a straightforward calculation shows that for all:

$$(v_1, v_2) \in T_xU_\alpha \oplus T_\lambda G = T_{(x,\lambda)}(U_\alpha \times G)$$

the map:

$$(\Psi_\mu^{-1})_*|_{(x,\lambda)} : T_{(x,\lambda)}(U_\alpha \times G) \to T_{s_{\mu,\lambda}(x)}(\pi^{-1}(U_\mu))$$

is explicitly given by:

$$(\Psi_\mu^{-1})_*|_{(x,\lambda)}(v_1) = (s_{\mu,\lambda})_*|_x(v_1) \tag{35}$$
$$(\Psi_\mu^{-1})_*|_{(x,\lambda)}(v_2) = (\theta|_\lambda(v_2))^\bullet|_{s_{\mu,\lambda}(x)} \tag{36}$$

A connection 1-form A on P can be pulled back via $s_{\mu,\lambda}$ to U_μ :

$$A_{\mu,\lambda} := s_{\mu,\lambda}^* A \in \Omega^1(U_\mu;\mathfrak{g})$$

Letting $A_\mu := s_\mu^* A = A_{\mu,e}$, a straightforward calculation shows that the Ad-equivariance of A implies:

$$A_{\mu,\lambda} = \mathrm{Ad}_\lambda^{-1} \circ A_\mu$$

Pulling back A to $U_\mu \times G$ via Ψ_μ^{-1}, another straightforward calculation using (35) and (36) shows that:

$$\begin{aligned} (\Psi_\mu^{-1})^* A &= A_{\mu,\lambda} + \theta \\ &= \mathrm{Ad}_{\lambda^{-1}} \circ A_\mu + \theta \end{aligned}$$

Principal bundles are a generalization of Lie groups. The manifold P generalizes a Lie group $\tilde{G}$. The structural group G generalizes a subgroup $G < \tilde{G}$. The right group action $\Phi : G \to \mathrm{Diff}(P)$ generalizes the right subgroup action $R : G \to \mathrm{Aut}(\tilde{G})$. For $\xi \in \mathfrak{g}$, the fundamental vector field $\xi^\bullet = \Phi_*|_e(\xi) \in \mathfrak{X}(P)$ generalizes the left-invariant vector field $\xi^\circ = R_*|_e(\xi) \in \mathfrak{X}(\tilde{G})$. The two properties (24) and (25) respectively generalize (21) and (23).

Now, it happens that sometimes on a principal bundle P we are not given a connection A but a metric g. Let's see what happens then.

5. A METRIC ON A PRINCIPAL BUNDLE

Let P be a G-principal bundle and let $V \subset TP$ be its vertical distribution. Suppose that P is endowed with a pseudo-Riemannian metric g such that the perpendicular distribution $H := V^{\perp}$ satisfies (29,30). Depending on g, the distribution H might or might not be G-invariant. Because V is G-invariant, a sufficient condition for H to be G-invariant is that g is G-invariant:

$$\Phi_\lambda^* g = g, \quad \forall \lambda \in G \tag{37}$$

This is the so-called *KK cylindrical condition* on g. The infinitesimal equivalent version of the cylindrical condition (37) is $\mathcal{L}_{\xi^\bullet} g = 0, \forall \xi \in \mathfrak{g}$.

May g be cylindrical or not, V and H do induce vertical and horizontal projections (33,34) and a unique $\mathfrak{g}$-valued differential 1-form $A \in \Omega^1(P;\mathfrak{g})$ on P that satisfies (24) and (28):

$$\begin{aligned} A(\xi^\bullet) &= \xi, \qquad \forall \xi \in \mathfrak{g} \\ H &= \ker(A) \end{aligned}$$

Such an A is not necessarily a connection form because it does not necessarily satisfy the Ad-equivariance property (25). In fact, A satisfies (25) if and only if the distribution $H = V^{\perp}$ is G-invariant. So, when g is cylindrical, A is a connection form.

May g be cylindrical or not, using a local trivialization Ψ_μ, we still have:

$$(\Psi_\mu^{-1})^* A = A_{\mu,\lambda} + \theta$$

Using the horizontal and the vertical projections, let's define two bilinear forms on P:

$$\begin{aligned} g_H(\cdot,\cdot) &:= g(\mathrm{hor}(\cdot), \mathrm{hor}(\cdot)) \\ g_V(\cdot,\cdot) &:= g(\mathrm{ver}(\cdot), \mathrm{ver}(\cdot)) \end{aligned}$$

Because H and V are perpendicular, the metric g decomposes as:

$$g = g_H + g_V \tag{38}$$

Let's define:

$$\begin{aligned} &K : P \to \mathfrak{g}^* \otimes \mathfrak{g}^*; a \mapsto K_a \\ &K_a(\xi_1,\xi_2) := g|_a(\xi_1^\bullet|_a, \xi_2^\bullet|_a), \quad \forall a \in P,\ \forall \xi_1,\xi_2 \in \mathfrak{g} \end{aligned}$$

This function K has nothing to do with the tensor K_{ij} defined in (14). The letter K is here chosen so that in certain circumstances it equals the bilinear Killing form K on $\mathfrak{g}$ defined as $\mathrm{Tr}(\mathrm{ad}_{\xi_1} \circ \mathrm{ad}_{\xi_2})$. Using (33), we have:

$$g_V(\cdot,\cdot) = K(A(\cdot), A(\cdot)) \tag{39}$$

Hence, (38) becomes:

$$g = g_H + K(A,A) \tag{40}$$

To sum it up, the group action Φ and the metric g on P induce all these things on P: H, V, A, g_H, g_V, K.

Let's choose a local trivialization Ψ_μ. To it corresponds a family of local trivializing sections $s_{\mu,\lambda}$. Via $s_{\mu,\lambda}$, let's pull back all these things down to U_μ:

$$\begin{aligned} A_{\mu,\lambda} &:= s_{\mu,\lambda}^* A \\ K_{\mu,\lambda} &:= s_{\mu,\lambda}^* K \\ g_{H,\mu,\lambda} &:= s_{\mu,\lambda}^* g_H \\ g_{V,\mu,\lambda} &:= s_{\mu,\lambda}^* g_V = K_{\mu,\lambda}(A_{\mu,\lambda}, A_{\mu,\lambda}) \\ g_{\mu,\lambda} &:= s_{\mu,\lambda}^* g = g_{H,\mu,\lambda} + g_{V,\mu,\lambda} \end{aligned}$$

The metric g pulled back to $U_\mu \times G$ is explicitly:

$$\begin{aligned} (\Psi_\mu^{-1})^* g &= (\Psi_\mu^{-1})^* g_H + (\Psi_\mu^{-1})^* g_V \\ &= g_{H,\mu,\lambda} + K_{\mu,\lambda}(\theta + A_{\mu,\lambda}, \theta + A_{\mu,\lambda}) \end{aligned}$$

If one supposes that g is cylindrical and supposes that K is Ad-invariant, then the metric (40) is a *Kaluza-Klein metric* with structural group G. The structural groups G we are interested is $(\mathbb{R},+)$.

6. A SPECIFIC $(\mathbb{R},+)$-PRINCIPAL BUNDLE

Let (Q,g) be a $(+,-,-,-,-)$ pseudo-Riemannian manifold endowed with a spacelike vector field $v \in \mathfrak{X}(Q)$. Assume that the vector field v defines a flow:

$$\begin{aligned} \Phi : (\mathbb{R},+) &\to \mathrm{Diff}(Q) \\ \lambda &\mapsto \Phi_\lambda \end{aligned}$$

that makes Q a $(\mathbb{R},+)$-principal bundle over $\tilde{Q} := Q/\mathbb{R}$. The group $(\mathbb{R},+)$ can be replaced by U(1) at wish. Consider a single coordinate x^4 on the 1-dimensional Lie group $(\mathbb{R},+)$. Let $\partial_4 := \partial/\partial x^4$. Doing so, $v = \partial_4^\bullet$ is a fundamental vector field on Q. The Maurer-Cartan 1-form on $(\mathbb{R},+)$ is $\theta = dx^4 \otimes \partial_4$. The metric g and the vector field v induce two distributions V and H on Q pointwise defined at each $q \in Q$ as:

$$\begin{aligned} V_q &:= \mathbb{R}\langle v_q \rangle \subset T_q Q \\ H_q &:= V_q^{\perp} \subset T_q Q \end{aligned}$$

Let's assume that g is cylindrical, i.e. let's assume that v is a Killing vector field of g:

$$\mathcal{L}_v g = 0$$

The metric g and the vector field v induce a Lie algebra valued differential 1-form form A on Q:

$$A := \frac{v^\flat}{\|v\|_g^2} \otimes \partial_4 \tag{41}$$

This differential 1-form satisfies the axioms (24,25) of a connection 1-form. Let's fix a local trivialization Ψ_μ

of Q and let $s_{\mu,\lambda}$ be its corresponding family of local trivializing sections. A straightforward calculation using the formulas of §5 shows that the pulled back metric $(\Psi_\mu^{-1})^* g$ on $U_\mu \times \mathbb{R}$ looks like this:

$$(\Psi_\mu^{-1})^* g = \tilde{g} - h^2 \beta \otimes \beta \tag{42}$$

Here, the function $h : Q \to \mathbb{R}_{>0}$, the differential 1-forms $\tilde{\beta}$ and β and the bilinear form $\tilde{g}$ are defined as:

$$h^2 = -K(v,v) = -\|v\|_g^2 \tag{43}$$
$$\tilde{\beta} = dx^4 \circ A_{\mu,\lambda} \tag{44}$$
$$\beta = dx^4 \circ (A_{\mu,\lambda} + \theta) = \tilde{\beta} + dx^4 \tag{45}$$
$$\tilde{g} = g_{H,\mu,\lambda} \tag{46}$$

The minus signs in (42,43) are because v is spacelike. The bilinear form $\tilde{g}$ is a $(+,-,-,-)$ metric on U_μ. Even if (42) looks like nothing important, it is in fact the usual shape of a standard 5D KK metric:

$$\left[(\Psi_\mu^{-1})^* g\right]_{ij} = \begin{bmatrix} \tilde{g}_{ij} - h^2\tilde{\beta}_i\tilde{\beta}_i & -h^2\tilde{\beta}_i \\ -h^2\tilde{\beta}_j & -h^2 \end{bmatrix}$$

For simplicity of the following presentation, I will assume that the local trivialization Ψ_μ is a global trivialization:

$$\Psi_\mu : Q \to \tilde{Q} \times \mathbb{R}$$

The equality (42) is now global on Q. We can simplify things even further by assuming that Q is not only diffeomorphic to $\tilde{Q} \times \mathbb{R}$ via Ψ_μ, but that it is equal to it:

$$Q = \tilde{Q} \times \mathbb{R}$$

Doing so, we can forget Ψ_μ and assume that the metric g on Q has this shape:

$$g = \tilde{g} - h^2 \beta \otimes \beta \tag{47}$$

7. KALUZA-KLEIN THEORY

Consider a $(+,-,-,-,-)$ cylindrical metric g on the 5D space $Q = \tilde{Q} \times \mathbb{R}$ that decomposes as (47):

$$g = \tilde{g} - h^2 \beta \otimes \beta \tag{48}$$

Here, h is a $\mathbb{R}_{>0}$-valued function, $\tilde{\beta} = \sum_{i\neq 4} \tilde{\beta}_i dx^i$ is a differential 1-form without dx^4, $\beta = \tilde{\beta} + dx^4$ is a differential 1-form and $\tilde{g}$ is a bilinear form without dx^4. Explicitly:

$$\begin{aligned} \tilde{g}(\partial_4, \cdot) &= \tilde{g}(\cdot, \partial_4) = 0 \\ \partial_4 \tilde{g}_{ij} &= 0 \\ \partial_4 h &= 0 \\ \tilde{\beta}(\partial_4) &= 0 \\ \partial_4 \tilde{\beta}_i &= 0 \end{aligned}$$

In KK theory, the field h is called the *dilaton field*. The indices of $\tilde{\beta}_i$ and of $\tilde{g}_{ij}$ range over $i,j = 0,...,3$. Let $\tilde{\beta}^i := \tilde{g}^{ij}\tilde{\beta}_j$ and $\|\tilde{\beta}\|_{\tilde{g}}^2 := \tilde{g}^{ij}\tilde{\beta}_i\tilde{\beta}_j$. The components g_{ij} of the metric g and the components g^{ij} of the inverse matrix $[g^{ij}] = [g_{ij}]^{-1}$ are:

$$g_{ij} = \begin{cases} \tilde{g}_{ij} - h^2\tilde{\beta}_i\tilde{\beta}_j & \text{for } i \neq 4, j \neq 4 \\ -h^2\tilde{\beta}_j & \text{for } i = 4,\ j \neq 4 \\ -h^2\tilde{\beta}_i & \text{for } i \neq 4,\ j = 4 \\ -h^2 & \text{for } i = 4,\ j = 4 \end{cases}$$

$$g^{ij} = \begin{cases} \tilde{g}^{ij} & \text{for } i \neq 4,\ j \neq 4 \\ -\tilde{\beta}^j & \text{for } i = 4,\ j \neq 4 \\ -\tilde{\beta}^i & \text{for } i \neq 4,\ j = 4 \\ \|\tilde{\beta}\|_{\tilde{g}}^2 - h^{-2} & \text{for } i = 4,\ j = 4 \end{cases}$$

A straightforward calculation shows that, indeed, we have $\delta^i_j = g^{ik}g_{kj}$. For $i,j,k,l \neq 4$, let:

$$\begin{aligned} \omega &:= \sum_{i,j\neq 4} \omega_{ij} dx^i \otimes dx^j = \sum_{i,j\neq 4} (\partial_i\tilde{\beta}_j - \partial_j\tilde{\beta}_i) dx^i \otimes dx^j \\ \|\omega\|_{\tilde{g}}^2 &:= \tilde{g}^{ik}\tilde{g}^{jl}\omega_{ij}\omega_{kl} \\ \tilde{\nabla}_i\omega_{jk} &:= \partial_i\omega_{jk} - \tilde{\Gamma}^l_{ij}\omega_{lk} - \tilde{\Gamma}^l_{ik}\omega_{jl} \end{aligned}$$

The following Christoffel symbols, Ricci curvature, scalar curvature, Einstein tensor and unitary 5D volume form are given by reformulating those computed in (Williams 2015). For $i,j,k,l \neq 4$, the Christoffel symbols Γ^k_{ij} of g are given as follow:

$$\begin{aligned} \Gamma^k_{ij} &= \tilde{\Gamma}^k_{ij} - \frac{1}{2}\tilde{g}^{kl}(\tilde{\beta}_j h^2\omega_{il} + \tilde{\beta}_i h^2 \omega_{jl} - 2\tilde{\beta}_i\tilde{\beta}_j h\partial_l h) \\ \Gamma^k_{i4} &= \Gamma^k_{4i} = -\frac{1}{2}\tilde{g}^{kl}(h^2\omega_{il} - 2\tilde{\beta}_i h \partial_l h) \\ \Gamma^k_{44} &= \tilde{g}^{kl} h \partial_l h \\ \Gamma^4_{ij} &= -\tilde{\beta}_l\tilde{\Gamma}^l_{ij} - \frac{1}{2}\tilde{\beta}^l\tilde{\beta}_j h^2\omega_{li} - \frac{1}{2}\tilde{\beta}_i\tilde{\beta}^l h^2\omega_{lj} \\ &\quad + \frac{1}{2}(\partial_i\tilde{\beta}_j + \partial_j\tilde{\beta}_i) + \tilde{\beta}_j\partial_i \ln h + \tilde{\beta}_i \partial_j \ln h \\ &\quad - \tilde{\beta}_i\tilde{\beta}_j\tilde{\beta}^l h\partial_l h \\ \Gamma^4_{i4} &= \Gamma^4_{4i} = -\frac{1}{2}\tilde{\beta}^l h^2\omega_{li} - \tilde{\beta}_i\tilde{\beta}^l h \partial_l h + \partial_i \ln h \\ \Gamma^4_{44} &= -\tilde{\beta}^l h\partial_l h \end{aligned}$$

For $i,j,k,l \neq 4$, the Ricci curvature of g is:

$$\begin{aligned} R_{ij} &= \tilde{R}_{ij} + \frac{1}{2}h^2\tilde{g}^{kl}\omega_{ik}\omega_{jl} - h^{-1}\tilde{H}_{ij}(h) \\ &\quad - \tilde{\beta}_i\tilde{\beta}_j R_{44} + \tilde{\beta}_i R_{j4} + \tilde{\beta}_j R_{i4} \\ R_{i4} &= -\frac{1}{2}h^2\tilde{g}^{jk}\tilde{\nabla}_j\omega_{ik} - \frac{3}{2}\tilde{g}^{jk}h(\partial_k h)\omega_{ij} + \tilde{\beta}_i R_{44} \\ R_{44} &= h\tilde{\Box}h + \frac{1}{4}h^4\|\omega\|_{\tilde{g}}^2 \end{aligned}$$

The scalar curvature of g is:

$$R = \tilde{R} - 2h^{-1}\tilde{\Box}h + \frac{1}{4}h^2\|\omega\|^2_{\tilde{g}} \tag{49}$$

For $i,j,k,l \neq 4$, the Einstein tensor of g is:

$$\begin{aligned}
G_{44} &= R_{44} + \frac{1}{2}h^2 R \\
&= \frac{1}{2}h^2\tilde{R} + \frac{3}{8}h^4\|\omega\|^2_{\tilde{g}} \\
G_{4i} &= R_{4i} + \frac{1}{2}h^2\tilde{\beta}_i R \\
&= \tilde{\beta}_i G_{44} - \frac{1}{2}h^2\tilde{g}^{kl}\tilde{\nabla}_l\omega_{ik} - \frac{3}{2}\omega_{ik}h\tilde{\partial}^k h \\
G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\
&= \tilde{G}_{ij} - \tilde{\beta}_i\tilde{\beta}_j G_{44} + \tilde{\beta}_i G_{j4} + \tilde{\beta}_j G_{i4} \\
&\quad + \frac{1}{2}h^2(\tilde{g}^{kl}\omega_{ik}\omega_{jl} - \frac{1}{4}\tilde{g}_{ij}\|\omega\|^2_{\tilde{g}}) \\
&\quad - \frac{1}{h}(\tilde{H}_{ij}h - \tilde{g}_{ij}\tilde{\Box}h)
\end{aligned}$$

The 5D volume form Ω_g of g is related to the 4D volume form $\Omega_{\tilde{g}}$ of $\tilde{g}$ as:

$$\Omega_g = h\Omega_{\tilde{g}} \wedge dx^4 \tag{50}$$

Now that we have all the KK formulas at our disposal, we can get the 4D EFE via the 5D HE action integral. Here I follow mostly (Williams 2015) and (Coquereaux & Esposito-Farese 1990). Substituting the scalar curvature (49) and the 5D volume form (50) of the x^4-independent metric (48) in the 5D HE action integral (11) on $Q = \tilde{Q} \times [x^4_0, x^4_1]$, we get:

$$\begin{aligned}
S_{\mathrm{HE}}[g] &= \frac{1}{2\kappa}\int_Q R\Omega_g \\
&= \frac{1}{2\kappa}\int_Q \left(\tilde{R} - 2h^{-1}\tilde{\Box}h + \frac{1}{4}h^2\|\omega\|^2_{\tilde{g}}\right) h\Omega_{\tilde{g}} \wedge dx^4 \\
&= \frac{1}{2\kappa}(x^4_1 - x^4_0)\int_{\tilde{Q}} \left(h\tilde{R} - 2\tilde{\Box}h + \frac{1}{4}h^3\|\omega\|^2_{\tilde{g}}\right)\Omega_{\tilde{g}}
\end{aligned}$$

Because $\tilde{\Box}h = \widetilde{\mathrm{Div}}(\tilde{\nabla}h)$ is a divergence term, we can discard it from the integral. Because the numerical value of the multiplicative constant $x^4_1 - x^4_0$ does not alter the EL equations, we can assume that it equals 1. Doing so, we get:

$$S_{\mathrm{HE}}[\tilde{g}, h, \tilde{\beta}] = \frac{1}{2\kappa}\int_{\tilde{Q}} \left(h\tilde{R} + \frac{1}{4}h^3\|\omega\|^2_{\tilde{g}}\right)\Omega_{\tilde{g}} \tag{51}$$

The usual 4D HE action integral does not have such a coupling between $\tilde{R}$ and h. As is usually done in KK theory, we can get rid of this coupling via a conformal transformation. More precisely, the needed conformal transformation is:

$$\tilde{g} = e^{2\varphi}\hat{g} \tag{52}$$

for $h = e^{-2\varphi}$, i.e. $\tilde{g} = h^{-1}\hat{g}$. Doing so, using the conformal transformation formulas in §2, we get:

$$\begin{aligned}
\tilde{g}_{ij} &= e^{2\varphi}\hat{g}_{ij} && (53)\\
\tilde{g}^{ij} &= e^{-2\varphi}\hat{g}^{ij} && (54)\\
\tilde{\Gamma}^k_{ij} &= \hat{\Gamma}^k_{ij} + \delta^k_i\partial_j\varphi + \delta^k_j\partial_i\varphi - \hat{g}_{ij}\hat{g}^{kl}\partial_l\varphi && (55)\\
\tilde{R} &= e^{-2\varphi}(\hat{R} - 6\hat{\Box}\varphi - 6\|d\varphi\|^2_{\hat{g}}) && (56)\\
\Omega_{\tilde{g}} &= e^{4\varphi}\Omega_{\hat{g}} && (57)\\
e^{-2\varphi}\tilde{R}\Omega_{\tilde{g}} &= (\hat{R} - 6\hat{\Box}\varphi - 6\|d\varphi\|^2_{\hat{g}})\Omega_{\hat{g}} && (58)
\end{aligned}$$

Discarding the divergence term $\hat{\Box}\varphi = \widehat{\mathrm{Div}}(\hat{\nabla}\varphi)$, the 4D HE integral becomes:

$$S_{\mathrm{HE}}[\hat{g}, h, \tilde{\beta}] = \int_{\tilde{Q}} \left(\frac{1}{2\kappa}\hat{R} + L\right)\Omega_{\hat{g}} \tag{59}$$

where the Lagrangian density is:

$$L = \frac{1}{2\kappa}\left(-\frac{3}{2}\|d\ln h\|^2_{\hat{g}} + \frac{1}{4}h^3\|\omega\|^2_{\hat{g}}\right) \tag{60}$$

If we were to use $\sigma = h^{1/3}$ as in (Gross & Perry 1983) p.34, the Lagrangian density (60) would look like:

$$L = \frac{1}{2\kappa}\left(-\frac{1}{6}\|d\ln\sigma\|^2_{\hat{g}} + \frac{1}{4}\sigma\|\omega\|^2_{\hat{g}}\right) \tag{61}$$

Also, using $h = e^{-2\varphi}$ as above, the Lagrangian density (60) can be seen as:

$$L = \frac{1}{2\kappa}\left(-6\|d\varphi\|^2_{\hat{g}} + \frac{1}{4}e^{-6\varphi}\|\omega\|^2_{\hat{g}}\right) \tag{62}$$

Let's now look at the EL equations corresponding to the HE action integral (59). Via variations in $\hat{g}$, we get respectively the EFE, the trace-reversed EFE and the Einstein traced equation:

$$\begin{aligned}
\hat{G}_{ij} &= \kappa\hat{T}_{ij} && (63)\\
\hat{R}_{ij} &= \kappa\hat{K}_{ij} && (64)\\
\hat{R} &= -\kappa\hat{T} && (65)
\end{aligned}$$

Here, $\hat{T}_{ij}$, $\hat{T}$, and $\hat{K}_{ij}$ are:

$$\begin{aligned}
\hat{T}_{ij} &= \frac{3}{2\kappa}\left(-(\partial_i\ln h)(\partial_j\ln h) + \frac{1}{2}\hat{g}_{ij}\|d\ln h\|^2_{\hat{g}}\right) \\
&\quad + \frac{h^3}{2\kappa}\left(\hat{g}^{kl}\omega_{ik}\omega_{jl} - \frac{1}{4}\hat{g}_{ij}\|\omega\|^2_{\hat{g}}\right) && (66)\\
\hat{T} &= \frac{3}{2\kappa}\|d\ln h\|^2_{\hat{g}} && (67)\\
\hat{K}_{ij} &= -\frac{3}{2\kappa}(\partial_i\ln h)(\partial_j\ln h) \\
&\quad + \frac{h^3}{2\kappa}\left(\hat{g}^{kl}\omega_{ik}\omega_{jl} - \frac{1}{4}\hat{g}_{ij}\|\omega\|^2_{\hat{g}}\right) && (68)
\end{aligned}$$

The EL equation for variations in $\tilde{\beta}$ is:

$$0 = \hat{\delta}(h^3\omega) \tag{69}$$

where $\hat{\delta}$ is the de Rham codifferential for $\hat{g}$. When h is constant, (69) reads $\hat{\delta}\omega = 0$. Combined with $d\omega = d^2\tilde{\beta} = 0$, these two equations are Maxwell's equations. Thus, the KK Maxwellian field ω behaves like a Maxwellian field. Also, (69) can be equivalently written as:

$$0 = \hat{\nabla}_i(h^3\omega^{ij}) \tag{70}$$

The EL equation for variations in h is:

$$h\hat{\Box}h - \|dh\|^2_{\hat{g}} = -\frac{1}{4}h^5\|\omega\|^2_{\hat{g}} \tag{71}$$

Using this identity:

$$\hat{\Box}\ln h = h^{-1}\hat{\Box}h - \|d\ln h\|^2_{\hat{g}}$$

the EL equation (71) is equivalent to:

$$\hat{\Box}\ln h = -\frac{1}{4}h^3\|\omega\|^2_{\hat{g}} \tag{72}$$

8. FROM THE 5D WAVE EQUATION TO THE 4D EQUATIONS OF MOTION

Let $\pentagon$ be the Laplace-Beltrami-Souriau operator corresponding to the KK metric (48) on Q. Suppose that a WKB wave with constant amplitude $\psi = e^{iS/\hbar}$ satisfies the 5D Klein-Gordon (KG) equation:

$$\pentagon\psi = -\left(\frac{\mu c}{\hbar}\right)^2\psi \tag{73}$$

Here, μ is a real constant playing a role analogous to the mass m in the 4D KG equation. This equation (73) was considered in e.g. (Souriau 1962). Letting $p := dS$, the equation (73) is equivalent to:

$$(\mu c)^2 = \|p\|^2_g \tag{74}$$
$$0 = \pentagon S \tag{75}$$

From (74,75), one can show that the *light rays* of ψ, i.e. the gradient curves of S, are geodesics in (Q, g).

Having $\mu \neq 0$ in (73) is useful when one wants to interpret KK theory as a unification of electromagnetism and gravity. But, in our present scenario, we are not interested in electromagnetism, we only want to see the mass of a 4D KG wave as a fifth momentum of a massless 5D wave. Doing so, we take $\mu = 0$ and the 5D KG equation (73) becomes the 5D wave equation:

$$\pentagon\psi = 0 \tag{76}$$

Similarly, (74,75) respectively become:

$$0 = \|p\|^2_g \tag{77}$$
$$0 = \pentagon S \tag{78}$$

Thus, the light rays of the WKB wave ψ are lightlike geodesics in the 5D space (Q, g). These lightlike geodesics in the 5D space Q correspond to some curves in the 4D space $\tilde{Q}$. Let's take a look at the equations of motion of these 4D curves.

Let $m_{\mathrm{P}} := \sqrt{\hbar c/G}$ be the *Planck mass*, introduced here only to match physical units. Let $u := \nabla S/m_{\mathrm{P}}$, so that p and u are related as:

$$m_{\mathrm{P}}g(u, \cdot) = p \tag{79}$$

Explicitly, the components $p_i = \partial_i S$ of the differential 1-form $p = p_i dx^i$ and the components $u^i = \partial^i S/m_{\mathrm{P}}$, where $\partial^i := g^{ij}\partial_j$, of the vector field $u = u^i\partial_i$ are related via the KK metric (48) as:

$$p_i = m_{\mathrm{P}}g_{ij}u^j \tag{80}$$
$$u^i = m_{\mathrm{P}}^{-1}g^{ij}p_j \tag{81}$$

For $i \neq 4$, a straightforward calculation shows that (80,81) imply:

$$p_4/m_{\mathrm{P}} = -h^2(\tilde{\beta}_i u^i + u^4) \tag{82}$$
$$m_{\mathrm{P}}u^i = \tilde{g}^{ij}(p_j - p_4\tilde{\beta}_j) \tag{83}$$

Let $\gamma = \gamma(\tau)$ be a gradient curve of S, i.e. $\dot{\gamma} = u$ where the dot denotes $d/d\tau$. The curve γ is a lightlike geodesic in the 5D space (Q, g). Because $0 = \nabla g$, the geodesic equation can be written in two ways:

$$0 = \nabla_u u$$
$$0 = \nabla_u p$$

These two equations can be written in terms of components for $i, j, k = 0, \ldots, 4$ as:

$$0 = u^i\partial_i u^k + u^i u^j\Gamma^k_{ij}$$
$$0 = u^i\partial_i p_k - u^i p_j\Gamma^j_{ik}$$

or, equivalently, as:

$$\dot{u}^k = -u^i u^j\Gamma^k_{ij} \tag{84}$$
$$\dot{p}_k = m_{\mathrm{P}}u^i u^l g_{jl}\Gamma^j_{ik} \tag{85}$$

The two terms of interest are $\dot{p}_4$ and $\dot{u}^k$ for $k \neq 4$. First of all, (85) implies that p_4 is a constant of motion:

$$\begin{aligned}\dot{p}_4/m_{\mathrm{P}} &= u^i u^l g_{jl}\Gamma^j_{i4} \\ &= u^i u^l g_{jl}\left(\frac{1}{2}g^{jm}(\partial_i g_{m4} + \partial_4 g_{mi} - \partial_m g_{i4})\right) \\ &= \frac{1}{2}u^i u^l\delta^m_l(\partial_i g_{m4} - \partial_m g_{i4}) \\ &= \frac{1}{2}u^i u^l(\partial_i g_{l4} - \partial_l g_{i4}) \\ &= 0\end{aligned}$$

Another way to see that p_4 is constant along the geodesic γ is to recall that the geodesic flow on (Q,g) is the projection to Q of the Hamiltonian flow on the cotangent bundle T^*Q of the x^4-independent Hamiltonian $H = \frac{1}{2m_\mathrm{P}} g^{ij} p_i p_j$.

Remark that even if p_4 is a constant of motion, u^4 is not necessarily constant. However, when h is constant and when $\tilde{\beta}$ vanishes, then u^4 is constant.

Now, a straightforward deployment of the indices of the KK metric (48) and of its Christoffel symbols in the equation of motion (84) gives, for $i,j,k \neq 4$:

$$\dot{u}^k + \tilde{\Gamma}^k_{ij} u^i u^j = -\frac{p_4}{m_\mathrm{P}} \tilde{g}^{jk} \omega_{ij} u^i - \frac{p_4^2}{m_\mathrm{P}^2} h^{-3} \tilde{\partial}^k h \tag{86}$$

Here, $\tilde{\partial}^k := \tilde{g}^{kj}\partial_j$. Using the above conformal transformation (52), i.e. $\hat{g} = h\tilde{g}$ for $h = e^{-2\varphi}$, we can substitute (53,53,53) in (86). Doing so, we get the equation of motion for γ in terms of $\hat{g}$ instead of $\tilde{g}$:

$$\begin{aligned}\dot{u}^k + \hat{\Gamma}^k_{ij} u^i u^j =& -\frac{p_4}{m_\mathrm{P}} h \hat{g}^{jk} \omega_{ij} u^i - \frac{p_4^2}{m_\mathrm{P}^2} h^{-2} \hat{\partial}^k h \\ & -\frac{1}{2}(\hat{\partial}^k \ln h) \|u\|^2_{\hat{g}} + (\partial_i \ln h) u^i u^k\end{aligned} \tag{87}$$

Here, $\hat{\partial}^k := \hat{g}^{kj}\partial_j$.

Another straightforward calculation using the metric (48), the conformal transformation (52), the relationships (80,81) and the equalities (82,83) shows that the norm of p and u in terms of g, $\tilde{g}$ and $\hat{g}$ are related as:

$$\|p\|^2_g = \|p - p_4\tilde{\beta}\|^2_{\tilde{g}} - p_4^2/h^2 \tag{88}$$
$$\|p\|^2_g = h\|p - p_4\tilde{\beta}\|^2_{\hat{g}} - p_4^2/h^2 \tag{89}$$
$$\|u\|^2_g = \|u\|^2_{\tilde{g}} - p_4^2/(h^2 m_\mathrm{P}^2) \tag{90}$$
$$\|u\|^2_g = \|u\|^2_{\hat{g}}/h - p_4^2/(h^2 m_\mathrm{P}^2) \tag{91}$$

In our scenario (77,78), we have $\|u\|^2_g = \|p\|^2_g/m_\mathrm{P}^2 = 0$. So, (88,89,90,91) become:

$$\|p - p_4\tilde{\beta}\|^2_{\tilde{g}} = p_4^2/h^2 \tag{92}$$
$$\|p - p_4\tilde{\beta}\|^2_{\hat{g}} = p_4^2/h^3 \tag{93}$$
$$\|u\|^2_{\tilde{g}} = p_4^2/(h^2 m_\mathrm{P}^2) \tag{94}$$
$$\|u\|^2_{\hat{g}} = p_4^2/(h m_\mathrm{P}^2) \tag{95}$$

There are two possibilities: either $p_4 = 0$ either $p_4 \neq 0$. If $p_4 = 0$, these four terms (92,93,94,95) will be constant along the path γ. However, if $p_4 \neq 0$ and if h is not constant along γ, none of these four terms will be constant along γ.

If p_4 is not zero: Assuming $p_4 \neq 0$, we can re-parametrize γ by its *proper time*. More precisely, above we had $\gamma = \gamma(\tau)$ such that $d\gamma(\tau)/d\tau = \nabla S/m_\mathrm{P}$ for some parameter τ. Doing so, (95) reads:

$$\left\| \frac{d\gamma(\tau)}{d\tau} \right\|^2_{\hat{g}} = \frac{p_4^2}{h m_\mathrm{P}^2} \tag{96}$$

Let τ' be the proper time of γ for $\hat{g}$, i.e.:

$$\left\| \frac{d\gamma(\tau')}{d\tau'} \right\|^2_{\hat{g}} = c^2 \tag{97}$$

Assuming that $d\tau'/d\tau > 0$, a straightforward calculation using (96) and (97) shows that $\tau'(\tau)$ must satisfy:

$$\frac{d\tau'}{d\tau} = \frac{p_4}{h^{1/2} m_\mathrm{P} c} \tag{98}$$

Using the proper time parametrization τ', the equation of motion (87) becomes:

$$\begin{aligned}\frac{d^2\gamma^k}{d\tau'^2} + \hat{\Gamma}^k_{ij} \frac{d\gamma^i}{d\tau'} \frac{d\gamma^j}{d\tau'} =& -h^{3/2} c \hat{g}^{jk} \omega_{ij} \frac{d\gamma^i}{d\tau'} - \frac{3}{2} c^2 \hat{\partial}^k \ln h \\ & + \frac{3}{2} \frac{d\ln h}{d\tau'} \frac{d\gamma^k}{d\tau'}\end{aligned} \tag{99}$$

Remark that (99) does not depend on the auxiliary Planck mass m_P introduced above to match physical units in (79). We can re-parametrize once more. Let τ'' such that:

$$\frac{d\tau''}{d\tau'} = h^{3/2}$$

This is equivalent to take $d\tau''/d\tau = hp_4/(m_\mathrm{P} c)$. Using the parametrization τ'', the equation of motion (99) becomes more simply:

$$\frac{d^2\gamma^k}{d\tau''^2} + \hat{\Gamma}^k_{ij} \frac{d\gamma^i}{d\tau''} \frac{d\gamma^j}{d\tau''} = -c \hat{g}^{jk} \omega_{ij} \frac{d\gamma^i}{d\tau''} - \frac{1}{2} c^2 \hat{\partial}^k (h^{-3}) \tag{100}$$

This is the simplest way to write the equation of motion when $p_4 \neq 0$. In either parametrization τ, τ' or τ'', because $p_4 \neq 0$ and because of (95), the motion is timelike in $(\tilde{Q}, \hat{g})$. This is compatible with the fact that we want here to describe a massive particle.

If p_4 is zero: Assuming $p_4 = 0$, the equation of motion (87) in terms of τ is:

$$\frac{d^2\gamma^k}{d\tau^2} + \hat{\Gamma}^k_{ij} \frac{d\gamma^i}{d\tau} \frac{d\gamma^j}{d\tau} = \frac{d\ln h}{d\tau} \frac{d\gamma^k}{d\tau} \tag{101}$$

Because $p_4 = 0$, it is not possible to re-parametrize by τ' or τ'' as above (unless we want to use the proper time of a massive clock with $p_4 \neq 0$). We can, however, consider the re-parametrization τ''' such that:

$$\frac{d\tau'''}{d\tau} = ah$$

for some real constant $a \neq 0$. In this case, the equation of motion (101) becomes more simply the geodesic equation in the 4D space $(\tilde{Q}, \hat{g})$:

$$\frac{d^2\gamma^k}{d\tau'''^2} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i}{d\tau'''}\frac{d\gamma^j}{d\tau'''} = 0 \tag{102}$$

Because $p_4 = 0$ and because of (95), this geodesic is lightlike in $(\tilde{Q}, \hat{g})$. This is compatible with the fact that we want here to describe a massless particle.

Now that we have the equations of motion for both cases $p_4 \neq 0$ and $p_4 = 0$, there are two possible interpretations for the equations of motion (99) and (102):

1. If we interpret p_4 as being related to an electrical charge, then up to an acceleration due to h, the equation of motion (99) is the motion due to electromagnetic Lorentz force. This is the usual electromagnetic KK interpretation, originally motivated by an eventual unification of electromagnetism with gravity (Kaluza 1921), (Klein 1926).

2. If we interpret p_4 as being related to an effective mass, but not to an electrical charge, then (99) describes a motion due to a fifth force. This fifth force is due to the KK dilaton field h and to the KK Maxwellian field ω. The impact of ω on the equation of motion of a point particle is a new Lorentz force.

The second interpretation is the one taken in this present document. Before jumping in the *dark* consequences of the new hypothetical fifth force, let's superficially skim over the *luminous* electromagnetic interpretation of KK theory and see what goes wrong with it.

9. EM INTERPRETATION OF KK THEORY

The original motivation for the KK theory was to unite electromagnetism and gravity inside one tensor to rule them all. Suppose here that h is constant. Then, the HE action integral (59), the Einstein field equation (63), the energy-momentum tensor (66) and the equation of motion (99) respectively become:

$$S_{\mathrm{HE}}[\hat{g}, \tilde{\beta}] = \int_{\tilde{Q}} \left(\frac{1}{2\kappa}\hat{R} + \frac{h^3}{8\kappa}\|\omega\|^2_{\hat{g}}\right)\Omega_{\hat{g}} \tag{103}$$

$$\hat{G}_{ij} = \kappa\hat{T}_{ij} \tag{104}$$

$$\hat{T}_{ij} = \frac{h^3}{2\kappa}\left(\hat{g}^{kl}\omega_{ik}\omega_{jl} - \frac{1}{4}\hat{g}_{ij}\|\omega\|^2_{\hat{g}}\right) \tag{105}$$

$$\frac{d^2\gamma^k}{d\tau'^2} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i}{d\tau'}\frac{d\gamma^j}{d\tau'} = -h^{3/2}c\hat{g}^{jk}\omega_{ij}\frac{d\gamma^i}{d\tau'} \tag{106}$$

The corresponding EM equations with the EM Maxwell tensor $F_{ij} = \partial_i A_i - \partial_j A_j$ are respectively:

$$S_{\mathrm{HE}}[\hat{g}, A] = \int_{\tilde{Q}} \left(\frac{1}{2\kappa}\hat{R} - \frac{1}{4\mu_0}\|F\|^2_{\hat{g}}\right)\Omega_{\hat{g}} \tag{107}$$

$$\hat{G}_{ij} = \kappa\hat{T}_{ij} \tag{108}$$

$$\hat{T}_{ij} = -\frac{1}{\mu_0}\left(\hat{g}^{kl}F_{ik}F_{jl} - \frac{1}{4}\hat{g}_{ij}\|F\|^2_{\hat{g}}\right) \tag{109}$$

$$\frac{d^2\gamma^k}{d\tau'^2} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i}{d\tau'}\frac{d\gamma^j}{d\tau'} = \frac{q}{m}\hat{g}^{jk}F_{ij}\frac{d\gamma^i}{d\tau'} \tag{110}$$

There is a sign problem to identify (103,104,105,106) with (107,108,109,110). This sign problem can be lifted by changing the signature of g from $(+,-,-,-,-)$ to $(+,-,-,-,+)$, e.g. as is done in (Williams 2015). Let's close our eyes on this issue. Up to signs headaches, we must identify:

$$\frac{h^3}{8\kappa}\|\omega\|^2_{\hat{g}} = \frac{1}{4\mu_0}\|F\|^2_{\hat{g}} \tag{111}$$

$$h^{3/2}c\omega_{ij} = \frac{q}{m}F_{ij} \tag{112}$$

This suggests to take a constant k such that $F = k\omega$. The first equation (111) implies:

$$k = h^{3/2}\sqrt{\frac{\mu_0}{2\kappa}} \tag{113}$$

Using (113) in the second equation (112), we get:

$$m = \sqrt{\frac{q^2\mu_0}{2c^2\kappa}}$$

For q the electrical charge e of a positron we get:

$$m = \frac{\sqrt{\alpha}}{2}m_{\mathrm{P}} \tag{114}$$

where $\alpha = \frac{\mu_0}{4\pi}\frac{e^2c}{\hbar} \approx \frac{1}{137}$ is the *fine structure constant*. Hence, the mass (114) cannot be the mass of a particle (e.g. electron, nucleon, etc.) because m_{P} is $\approx 10^{20}$ too heavy.

Now, the idea of this present document is that the momentum in the fifth dimension does not represent an electric charge but mass only. Doing so, we do not need to rescale to $F = k\omega$ or change the signature of g. Let's stay with the $(+,-,-,-,-)$ metric (48) and keep ω.

10. MASS AS A MOMENTUM

Comparing (93):

$$\|p - p_4\tilde{\beta}\|^2_{\hat{g}} = p_4^2/h^3$$

to the classical definition of mass as the norm of a time-like 4-momentum:

$$\|p\|^2_{\hat{g}} = (mc)^2 \tag{115}$$

we get a relationship between the constant of motion p_4, the field h and the mass m:

$$p_4 = h^{3/2} mc \tag{116}$$

This equation (116) is the precise mathematical formulation of the *mass = momentum hypothesis* mentioned in §1. Here I assume it to hold true even when $\tilde{\beta} \neq 0$:

$$\|p - p_4 \tilde{\beta}\|^2_{\hat{g}} = (mc)^2 \tag{117}$$

so that mass can be seen as a coupling to the field $\tilde{\beta}$.

Remark that the relationship (116) gives rise to a *mass operator* $\hat{m}$:

$$\hat{m}c = h^{-3/2}\hat{p}_4 = -i\hbar h^{-3/2}\partial_4 \tag{118}$$

This mass operator generalizes the one with $h = 1$ defined in (Aubin-Cadot 2018). Because we are in a pseudo-Riemannian setting which is not necessarily a flat Minkowskian one, there should be a divergence term added in (118) as geometric quantization predicts (Sniatycki 1980), p.128. However, the mass operator is not our focus here. I will shortly get back to it in §15.

For a given p_4, the relationship (116) between m and h seems to imply that the dilaton field h should be related to the Higgs field. In the EM interpretation of KK theory, such a possible relationship between the dilaton and the Higgs was mentioned in e.g. (Witten 1981), p.425. In the present MMH interpretation of 5D KK theory, I will discuss this possible relationship between the dilaton and the Higgs in §14.

Because $h \neq 0$, we know that p_4 vanishes if and only if m vanishes. Hence, the two scenarios $p_4 \neq 0$ and $p_4 = 0$ in §8 were respectively the *massive* and the *massless* scenarios.

In the massive scenario $p_4 \neq 0$, the equation of motion (99) parametrized by the proper time τ' in the 4D spacetime $(\tilde{Q}, \hat{g})$ is independent of the precise numerical value of p_4. Because the mass m depends on the value of the field h, the *equivalence principle* of GR must be slightly reformulated so as to say that the motion due to $\hat{g}$, h and $\tilde{\beta}$ is independent of the precise numerical value of the constant of motion $p_4 \neq 0$.

In the massless scenario $p_4 = 0$, the motion in the 4D spacetime $(\tilde{Q}, \hat{g})$ is described by the geodesic equation (102):

$$\frac{d^2\gamma^k}{d\tau'''^2} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i}{d\tau'''}\frac{d\gamma^j}{d\tau'''} = 0$$

Thus, the motion of a massless particle is a lightlike geodesic which is independent of h and $\tilde{\beta}$. Hence, the fields h and $\tilde{\beta}$ do not affect the path of e.g. *light*. At best, h will affect the phase of a lightlike geodesic, not its 4D spacetime path.

Remark that there is a subtlety in the above mentioned *equivalence principle*. Although for $p_4 \neq 0$ the motion due to $\hat{g}$, $\tilde{\beta}$ and h does not depend on the precise numerical value of p_4, the motion does depend on the fact that $p_4 \neq 0$.

Now that we have the desired definition (116) of mass as a momentum and that we have the equations of motion in the 4D spacetime $(\tilde{Q}, \hat{g})$, we need to interpret the fifth force due to $\tilde{\beta}$ and h.

11. THE RESULTING FIFTH FORCE AND THE ROTATIONAL CURVE OF GALAXIES

In the above mentioned *equivalence principle*, there is a clear distinction between the motion of a massive point particle and that of a massless one. Massive particles are coupled to the fields h and $\tilde{\beta}$ while massless particles aren't. Because photons are massless, the fifth force is *invisible* (gravity is *visible* because it bends 4D lightlike geodesics). Because $\tilde{\beta}$ behaves like electromagnetism, while not being electromagnetism, we already have a good intuition of how $\tilde{\beta}$ behaves and how $\tilde{\beta}$ affects matter. The effect of $\tilde{\beta}$ on matter is a Lorentz-like force. The Lorentz force has two parts. Its first part is the "electric" part, which could be confused with usual Newtonian gravitation. The other part is the "magnetic" part, which could be confused with the gravitomagnetic frame-dragging *Lense–Thirring* effect due to a spinning body. However, being gravitational, the Lense-Thirring effect has an impact on 4D lightlike geodesics, while $\tilde{\beta}$ has no impact on them.

There is no observed fifth force at the quantum level, nor at the scale of a planetary system. Thus, the fifth force should be very weak. For example, its impact on the precession of Mercury's perihelion should fit inside the tiny relative error between observation and GR's predicted value. This suggests that the fifth force should manifest itself at scales at least as big as a galaxy or the cosmos.

Consider a poor man's point particle galaxy where all its mass M lies at the origin $(0, 0, 0)$. The module of the Newtonian gravitational acceleration of a non-relativistic test particle orbiting around the galaxy is $a = GM/r^2$. Assuming that the test particle follows a circular orbit, its orbital speed is $v(r) = \sqrt{GM/r}$. However, physical observations tells us that the observed orbital speed $v(r)$ is not $\propto 1/\sqrt{r}$ but often roughly $v(r) \propto$ *const.*. Even a rich man's thick disc-shaped galaxy does not solve this problem.

Dark matter is a hypothetical *matter* whose Newtonian gravitational effect solves the $v(r) \approx$ *const.* problem. However, dark matter particles were never detected. There are various theoretical alternatives to the dark matter hypothesis (Mannheim 2006). This is where the fifth force due to $\tilde{\beta}$ and h comes in.

First, let's look only at the effect of $\omega = d\tilde{\beta}$. Let's call the "magnetic" part of ω *dork magnetism* to distinguish it from conventional magnetism and also from the already defined *dark magnetism* (Jiménez & Maroto 2011). The speed of a massive non-relativistic test particle following a circular orbit due to a dork magnetic field $B_z \propto 1/r$ is $v(r) \approx const.$. Thus, the rotational curve of a galaxy could be explained in terms of dork magnetism. Let's call this the *dork magnetic explanation of the rotational curves of galaxies* (DMEOTRCOG). Because the field ω behaves like a Maxwellian field, we know how to "activate" such a dork magnetic field $B_z \propto 1/r$. All is needed is a strong enough spin density inside the galaxy. Such a spin density would come from all the spinning constituents of the galaxy, e.g. planetary systems, stars, planets, fidget spinners, etc.. Appart from a needed fifth dimension, such a DMEOTRCOG is somewhat better than its classical magnetic explanation (Battaner & al. 1992) because it affects all the massive constituents of the galaxy, not just its ionized constituents. However, such a DMEOTRCOG would probably be dismantled by e.g. (Persic & Salucci 1993) regarding the dynamic of two neighbouring galaxies. Moreover, because dork magnetism does not affect the path of light, the DMEOTRCOG is incompatible with the empirical observation of dark matter filaments (Epps & Hudson 2017). Also, it might be easier to invoke merely Lense-Thirring effect (Bruskiewich 2001) instead of dork magnetism to explain the rotational curve of a galaxy composed of, say, one mole of fidget spinners. To sum it up, $\tilde{\beta}$ might not be the best candidate to explain the rotational curve of galaxies.

Could h be taken responsible for the rotational curve of galaxies? The non-relativistic acceleration of a point particle due to h on a flat spacetime $(\tilde{Q}, \hat{g}) = (\mathbb{R}^4, \eta)$ with $\tilde{\beta} = 0$ is given by (99):

$$\frac{d^2\gamma^k}{d\tau'^2} = -\frac{3}{2}c^2\hat{\partial}^k \ln h + \frac{3}{2}\frac{d\ln h}{d\tau'}\frac{d\gamma^k}{d\tau'} \tag{119}$$

Assuming that $h = h(r)$ and that the orbit is circular, h is constant along the orbit so that (119) becomes:

$$\frac{d^2\gamma^k}{d\tau'^2} = -\frac{3}{2}c^2\hat{\partial}^k \ln h \tag{120}$$

Here, $\hat{\partial}^k = \eta^{kj}\partial_j$. The radial acceleration is then:

$$a \propto \partial_k \ln h$$

Thus, to get the galactic orbital speed $v(r) \propto const.$ we must take $h(r) \propto 1/r$. Doing so, according to (116), the mass of particles with a given p_4 will vary inside the galaxy along the radius r:

$$m(r) \propto h(r)^{-3/2} \propto r^{3/2}$$

Hence, particles are heavier far from the center of the galaxy. In particular, this should imply that the Higgs varies quite a lot at the scale of a galaxy. However, once again, explaining the rotational curve of galaxies in terms of h is not compatible with the observation of dark matter filaments. Indeed, h does not affect the path of light.

Now that h and $\tilde{\beta}$ are failing to explain the rotational curves of galaxies, mostly because of dark matter filaments, let's forget about dark matter. The never observed point particle dynamic predicted by h in (99) is, at least in an EM interpretation of KK theory, problematic (Gegenberg & Kunstatter 1984), (Kovacs 1984). For this reason, I suggest here a mathematical twist to the usual KK theory to get rid of it.

12. GETTING RID OF THE DILATON FIELD

In usual 5D KK theory lies a fixed background spacelike Killing vector field v of (Q, g). The dilaton field $h : Q \to \mathbb{R}_{>0}$ was defined in (43) as $h^2 := -\|v\|_g^2$. When doing variations of g in the HE action integral, the dilaton h varies too. Doing so, the dilaton is part of the 5D EFE and also contributes to the derived 4D Lorentz force coming from the 5D geodesic equation. As mentioned at the end of §11, the point particle dynamic due to this dilaton is known to be problematic. I will now suggest a way to get rid of the dilaton field.

Let Q be a smooth real 5-manifold. Let TQ be the *tangent bundle* of Q and let $\mathrm{Fr}(Q)$ be its *tangent frame bundle*. Explicitly, at each $x \in Q$ we have:

$$\mathrm{Fr}_x(Q) := \{f \in \mathrm{Isomorphisms}(\mathbb{R}^5; T_xQ)\}$$

The frame bundle $\mathrm{Fr}(Q)$ is a right $\mathrm{GL}(5;\mathbb{R})$-principal bundle on Q. The right group action of $\lambda \in \mathrm{GL}(5;\mathbb{R})$ on $f \in \mathrm{Fr}(Q)$ is the composition $f \circ \lambda$. Let (e_i) be the canonical basis of $\mathbb{R}^5$:

$$e_0 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, e_1 = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, e_4 = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

Let (e^i) be the canonical dual basis defined as $e^i(e_j) := \delta^i_j$. On $\mathbb{R}^5$ lies a canonical 5D Minkowski metric:

$$\eta = e^0 \otimes e^0 - e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 \tag{121}$$

Choosing a $(+,-,-,-,-)$ metric g on Q is equivalent to do a $\mathrm{O}(1,4)$ structural reduction $\mathrm{Fr}^{\mathrm{O}(1,4)}(Q) \subset \mathrm{Fr}(Q)$ of the tangent frame bundle via the correspondence:

$$\mathrm{Fr}_x^{\mathrm{O}(1,4)}(Q) = \{f \in \mathrm{Fr}_x(Q) : g_x(f(\cdot), f(\cdot)) = \eta(\cdot,\cdot)\}$$

The bundle $\mathrm{Fr}^{\mathrm{O}(1,4)}(Q)$ is a right $\mathrm{O}(1,4)$-principal bundle on Q.

Now, instead of defining a background spacelike vector field v on Q independently of g, we can define v and g at the exact same time. Consider this subgroup inclusion:

$$\mathrm{O}(1,3) = \begin{bmatrix} \mathrm{O}(1,3) & 0 \\ 0 & 1 \end{bmatrix} \subset \mathrm{O}(1,4) \subset \mathrm{GL}(5,\mathbb{R})$$

Doing so, $e_4 \in \mathbb{R}^5$ is $\mathrm{O}(1,3)$-invariant. Let's choose some $\mathrm{O}(1,3)$ structural reduction $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q) \subset \mathrm{Fr}(Q)$. The bundle $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q)$ is a right $\mathrm{O}(1,3)$-principal bundle on Q. This structural reduction induces both a $(+,-,-,-,-)$ metric g and a spacelike vector field v on Q pointwise defined as:

$$g_x(\cdot,\cdot) := \eta(f^{-1}(\cdot) f^{-1}(\cdot)) \tag{122}$$
$$v_x := f(e_4) \tag{123}$$

for any $f \in \mathrm{Fr}_x^{\mathrm{O}(1,3)}(Q)$. These definitions do not depend on the choice of $f \in \mathrm{Fr}_x^{\mathrm{O}(1,3)}(Q)$ because e_4 and η are both $\mathrm{O}(1,3)$-invariant. The structural reduction $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q)$ induces also a right $\mathrm{O}(1,4)$-principal bundle on Q:

$$\mathrm{Fr}_x^{\mathrm{O}(1,4)}(Q) := \{f \circ \lambda : f \in \mathrm{Fr}_x^{\mathrm{O}(1,3)}(Q), \lambda \in \mathrm{O}(1,4)\}$$

This last $\mathrm{O}(1,4)$ bundle on Q also corresponds to the metric g on Q. Now, something magical happens. The dilaton field h is automatically set to be constant and equal to 1. Indeed, for any $f \in \mathrm{Fr}_x^{\mathrm{O}(1,3)}$ we have:

$$\begin{aligned} h(x)^2 &= -\|v_x\|_g^2 \\ &= -g_x(v_x, v_x) \\ &= -\eta(f^{-1}(f(e_4)), f^{-1}(f(e_4))) \\ &= -\eta(e_4, e_4) \\ &= 1 \end{aligned}$$

This way of seeing things is different than in usual KK theory. In usual KK theory, variations of the metric g are done with a fixed background v. This amounts to look at variations of a $\mathrm{O}(1,4)$ structural reduction of $\mathrm{Fr}(Q)$ with a fixed background v. The way I presented here consists instead to look at variations of a $\mathrm{O}(1,3)$ structural reduction of $\mathrm{Fr}(Q)$. In particular, v becomes part of the variational principle. However, I do not have an action integral defined on such $\mathrm{O}(1,3)$ structural reductions at hand. For this reason, I will stay with the usual KK theory where v and g are kept independent. Thus, there is a non-constant dilaton field h. Doing so, choosing g amounts to do a first structural reduction $\mathrm{Fr}^{\mathrm{O}(1,4)}(Q) \subset \mathrm{Fr}(Q)$. Then, reducing further $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q) \subset \mathrm{Fr}^{\mathrm{O}(1,4)}(Q)$ amounts to define a spacelike vector field u such that $\|u\|_g^2 = -1$ defined as:

$$u = f(e_4) \tag{124}$$

for any section f of the bundle $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q)$. Then, the spacelike Killing vector field v on (Q, g) can be seen as being:

$$v = hu \tag{125}$$

for a $\mathbb{R}_{>0}$-valued function h on Q. Doing so, the relationship between the spacelike vector field v and the dilaton h is the same as (43):

$$h^2 = -\|v\|_g^2 \tag{126}$$

Remark that choosing the unitary vector field u, i.e. choosing the reduction $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q) \subset \mathrm{Fr}^{\mathrm{O}(1,4)}(Q)$ is equivalent to choose a section of the *de Sitter bundle* $\mathrm{Fr}^{\mathrm{O}(1,4)}(Q)/\mathrm{O}(1,3)$ over Q whose typical fiber is the *de Sitter space* $\mathrm{O}(1,4)/\mathrm{O}(1,3) = S^3 \times \mathbb{R}$.

Now, let's investigate similarities between the dilaton field h and the Higgs field.

13. THE MASS OF THE DILATON FIELD

The Higgs field ϕ is a $\mathbb{C}^2$-valued field that gives mass to some gauge fields and some matter fields (Hamilton 2015). This given mass is proportional to $\|\phi\|_{\mathrm{E}} := \sqrt{\phi^\dagger \phi}$. The standard Euclidean norm $\|\cdot\|_{\mathrm{E}}$ on $\mathbb{C}^2$ can be seen as coming from the standard $(+,+,+,+)$ Euclidean scalar product $\langle\cdot,\cdot\rangle_{\mathrm{E}}$ on $\mathbb{R}^4 = \mathbb{R}\langle e_1, e_2, e_3, e_4\rangle$:

$$\langle\cdot,\cdot\rangle_{\mathrm{E}} = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4 \tag{127}$$

Because of the so-called *mexican hat potential*:

$$V(\Phi) = -\frac{1}{2}\mu\|\phi\|_{\mathrm{E}}^2 + \frac{1}{2}\lambda\|\phi\|_{\mathrm{E}}^4$$

for $\mu, \lambda > 0$, the vacuum expectation value for $\|\phi\|_{\mathrm{E}}$ is a non-vanishing positive constant $r_{\mathrm{H}} := \sqrt{\mu/2\lambda} > 0$. Doing so, ϕ is expected to take values in a standard 3-sphere $S^3 \subset \mathbb{R}^4$ of radius r_{H}. After an appropriate $\mathrm{U}(1) \times \mathrm{SU}(2)$ gauge transformation, the $(\mathbb{C}^2 = \mathbb{R}^4)$-valued Higgs field ϕ can be set to look like:

$$\phi = \|\phi\|_{\mathrm{E}} e_4 \tag{128}$$

When the field ϕ is excited, the module $\|\phi\|_{\mathrm{E}}$ can fluctuate around its expected radius r_{H}. This little fluctuation of the Higgs field around its vacuum expectation value is the *Higgs boson*. The field equation for the Higgs boson is a KG equation. This KG equation follows from approximating the field equation of ϕ around its vacuum. The mass term in the Higgs boson's KG equation is the mass of the Higgs boson.

We can follow a similar argument with the dilaton field h. The field equation for the field h is given by (72):

$$\hat{\Box}\ln h = -\frac{1}{4}h^3\|\omega\|^2_{\hat{g}}$$

Because h is $\mathbb{R}_{>0}$-valued, we can write h as $h = e^\nu$ for some $\mathbb{R}$-valued function ν. Thus, (72) becomes:

$$\hat{\Box}\nu = -\frac{1}{4}e^{3\nu}\|\omega\|^2_{\hat{g}} \tag{129}$$

We do not have a potential for ν that sets its vacuum expectation value. Such an eventual potential would correspond, for example, to a potential that sets the vacuum expectation value of the metric g to a Minkowski metric η. Let's nevertheless suppose that the vacuum expectation value of ν is some function ν_0 that satisfies (129). Let's consider a small fluctuation $\nu_1 \ll 1$ so that $\nu = \nu_0 + \nu_1$. Then, (129) implies that the small fluctuation ν_1 satisfies:

$$\hat{\Box}\nu_1 \approx -\frac{3}{4}e^{3\nu_0}\|\omega\|^2_{\hat{g}}\nu_1 \tag{130}$$

Comparing (130) to the KG equation:

$$\hat{\Box}\nu_1 = -\left(\frac{mc}{\hbar}\right)^2\nu_1$$

we see that the fluctuation ν_1 has a mass m given by:

$$m = \frac{\sqrt{3}\hbar}{2c}e^{(3/2)\nu_0}\|\omega\|_{\hat{g}}$$

The mass term m here is not the mass of the dilaton $h = e^\nu$ but of the fluctuation ν_1. This mass is defined via the KG equation, not via the MMH. The MMH cannot define a mass for ν_1 because it would contradict the cylindrical hypothesis.

Now, let's dig further in the analogy between the dilaton and the Higgs.

14. THE DILATON AND THE HIGGS

The $(+,+,+,+)$ Euclidean space $(\mathbb{C}^2 = \mathbb{R}^4, \langle\cdot,\cdot\rangle_{\mathrm{E}})$ mentioned in §13 in which the Higgs field ϕ takes values can be seen as the Euclidean subspace of the $(+,-,-,-,-)$ space $(\mathbb{R}^5, \eta)$ mentioned in §12 via the subspace injection:

$$\begin{aligned}\mathbb{R}^4 &\to \mathbb{R}^5 \\ (x^1,...,x^4) &\mapsto (0,x^1,...,x^4)\end{aligned} \tag{131}$$

Doing so, the basis vector $e_4 = (0,0,0,1)$ of $\mathbb{R}^4$ is identified with the basis vector $e_4 = (0,0,0,0,1)$ of $\mathbb{R}^5$ and the bilinear forms (121) and (127) are related as:

$$\eta = e^0 \otimes e^0 - \langle\cdot,\cdot\rangle_{\mathbb{E}} \tag{132}$$

From (124,125,126) we can write the spacelike vector field v on (Q, g) as:

$$v = hu = \sqrt{-\|v\|^2_g}f(e_4) \tag{133}$$

where f is any section of $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q) \subset \mathrm{Fr}^{\mathrm{O}(1,4)}(Q)$. From (128,132), the evaluation of f on the Higgs field ϕ is also a spacelike vector field on Q:

$$f(\phi) = f(\|\phi\|_{\mathrm{E}}e_4) = \|\phi\|_{\mathrm{E}}f(e_4) = \sqrt{-\|\phi\|^2_\eta}f(e_4) \tag{134}$$

Comparing (133) and (134), it seems like h and $\|\phi\|_{\mathrm{E}}$ play a similar role as being the norm of some vectors. Thus, the relationship between the dilaton and the Higgs mentioned back in §10 regarding their influence on masses is not only scalar but also vectorial.

Now, one could complain because the Higgs, a spin 0 field, is now identified with a vector field and vector fields are known to be spin 1. But, there is no problem. Vector fields on $\tilde{Q} = Q/\mathbb{R}$ are spin 1 because of how the *Lorentz group* $\mathrm{O}(1,3)$ act on them. Because the Higgs is identified with $v = hf(e_4)$ on Q and because e_4 is invariant under the action of $\mathrm{O}(1,3) < \mathrm{O}(1,4)$, the Higgs has spin 0.

The identification (131) of the $(+,+,+,+)$ space $\mathbb{C}^2 = \mathbb{R}^4$ in which the Higgs lives and the $(+,-,-,-,-)$ space $\mathbb{R}^5$ does not simply relate the Higgs to the dilaton. It expresses the electroweak structural group $\mathrm{U}(1)\times\mathrm{SU}(2)$ as a subgroup of the structural group $\mathrm{GL}(5;\mathbb{R})$ of the tangent frame bundle $\mathrm{Fr}(Q)$:

$$\mathrm{U}(1)\times\mathrm{SU}(2) < \mathrm{O}(4) = \begin{bmatrix}1 & 0 \\ 0 & \mathrm{O}(4)\end{bmatrix} < \mathrm{O}(1,4) < \mathrm{GL}(5;\mathbb{R})$$

Let's define ten 5×5 real matrices:

$$K_1 = \begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix} \quad K_2 = \begin{bmatrix}0&0&1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$$K_3 = \begin{bmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{bmatrix} \quad K_4 = \begin{bmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{bmatrix}$$

$$J_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad J_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad J_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$J_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \qquad J_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

For $1 \leq i < j \leq 4$, the commutator of two *boosts* K_i is an infinitesimal rotation $[K_i, K_j] = J_{ij}$. Let's define four more 5×5 real matrices:

$$\begin{aligned} X_0 &:= -J_{13} - J_{24} \\ X_1 &:= -J_{23} - J_{14} \\ X_2 &:= J_{12} + J_{34} \\ X_3 &:= -J_{13} + J_{24} \end{aligned}$$

They satisfy:

$$\begin{aligned} [X_0, X_1] &= 0 \qquad & [X_1, X_2] &= -2X_3 \\ [X_0, X_2] &= 0 \qquad & [X_3, X_1] &= -2X_2 \\ [X_0, X_3] &= 0 \qquad & [X_2, X_3] &= -2X_1 \end{aligned}$$

Thus, we have:

$$\begin{aligned} \mathfrak{so}(1,4) &= \mathbb{R}\langle K_1, K_2, K_3, K_4, J_{12}, J_{13}, J_{14}, J_{23}, J_{24}, J_{34}\rangle \\ \mathfrak{so}(1,3) &= \mathbb{R}\langle K_1, K_2, K_3, J_{12}, J_{13}, J_{23}\rangle < \mathfrak{so}(1,4) \\ \mathfrak{so}(4) &= \mathbb{R}\langle J_{12}, J_{13}, J_{14}, J_{23}, J_{24}, J_{34}\rangle < \mathfrak{so}(1,4) \\ \mathfrak{u}(1) &= \mathbb{R}\langle X_0\rangle < \mathfrak{so}(4) \\ \mathfrak{su}(2) &= \mathbb{R}\langle X_1, X_2, X_3\rangle < \mathfrak{so}(4) \end{aligned}$$

Evaluating the basis elements of $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$ on the Higgs $\phi = \|\phi\|_{\mathrm{E}} e_4 \propto e_4$ we have:

$$\begin{aligned} X_0 e_4 &= -e_2 \\ X_1 e_4 &= -e_1 \\ X_2 e_4 &= e_3 \\ X_3 e_4 &= e_2 \end{aligned}$$

In particular, we have:

$$(X_0 + X_3)e_4 = -e_2 + e_2 = 0$$

Identifying electromagnetism with the diagonal element $X_0 + X_3 \in \mathfrak{u}(1) \oplus \mathfrak{su}(2)$, electromagnetism is not coupled to the Higgs $\phi \propto e_4$. Albeit being done in a $\mathbb{R}^5$ representation and not in a more common $\mathbb{C}^2$ one, this is the usual electroweak way of dealing with the fact that photons are massless. Now, despite the above relationship between the dilaton and the Higgs via the vector field v, the electroweak way to give mass via the Higgs is different than the above MMH way of seing mass as a momentum in the direction spanned by the vector field v. I do not have a remedy at hand to cure this discrepancy of geometrical mechanisms to define mass. Thus, let's move on.

We have a vector space split:

$$\mathfrak{so}(1,4) = \mathfrak{so}(1,3) \oplus \frac{\mathfrak{so}(1,4)}{\mathfrak{so}(1,3)} \tag{135}$$

where:

$$\frac{\mathfrak{so}(1,4)}{\mathfrak{so}(1,3)} = \mathbb{R}\langle K_4, J_{14}, J_{24}, J_{34}\rangle$$

This splitting is $\mathrm{Ad}(\mathrm{O}(1,3))$-invariant. Let's define:

$$\begin{aligned} \mathfrak{g} &:= \mathrm{Lie}(\mathrm{O}(1,4)) = \mathfrak{so}(1,4) \\ \mathfrak{h} &:= \mathrm{Lie}(\mathrm{O}(1,3)) = \mathfrak{so}(1,3) \\ \mathfrak{p} &:= \mathfrak{g}/\mathfrak{h} = \mathbb{R}^4 \end{aligned}$$

Thus, the splitting (135) becomes more simply:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{136}$$

In terms of algebra, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{p}] = \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{h}$. Via left multiplication of the above matrices by the 5×5 matrix $[\eta_{ij}] = \mathrm{Diag}(1, -1, -1, -1, -1)$ we can identify $\mathfrak{g} = \wedge^2\mathbb{R}^5$, $\mathfrak{h} = \wedge^2\mathbb{R}^4$ and $\mathfrak{p} = \wedge^1\mathbb{R}^4$. Thus, the splitting (136) can be seen as:

$$\wedge^2\mathbb{R}^5 = \wedge^2\mathbb{R}^4 \oplus \wedge^1\mathbb{R}^4 \tag{137}$$

The Hodge star operator $\star$ on $\wedge^2\mathbb{R}^4$, for $\mathbb{R}^4$ Minkowskian, satisfies $\star^2 = -1$ so that we have a natural complex structure J induced on the Lie algebra $\mathfrak{h}$.

Now, let's consider a connection form A on the $\mathrm{O}(1,4)$-principal bundle $\mathrm{Fr}^{\mathrm{O}(1,4)}(Q)$. This connection form A is, by definition, $\mathfrak{g}$-valued. Because of the vector space splitting (136), the connection form A splits as a sum of a $\mathfrak{h}$-valued 1-form and a $\mathfrak{p}$-valued 1-form:

$$A = A^{\mathfrak{h}} + A^{\mathfrak{p}}$$

Restricting these three 1-forms to the subbundle $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q)$, we get three 1-forms:

$$\alpha = \alpha^{\mathfrak{h}} + \alpha^{\mathfrak{p}}$$

The 1-form $\alpha^{\mathfrak{h}}$ is a connection form on $\mathrm{Fr}^{\mathrm{O}(1,3)}(Q)$ while the 1-form $\alpha^{\mathfrak{p}}$ is a basic 1-form that goes down to a $\mathfrak{p}$-bundle-valued 1-form. Such a decomposition was considered in (Wise 2007) to express the 4D HE action and its corresponding 4D EFE in terms of a Yang-Mills-ish theory via the above correspondence between algebra and n-forms. However, in the 4D context of (Wise 2007), the use of so$(1,4)$ seems a bit *ad hoc* while it is natural in a 5D KK context.

Thus, it seems like a 5D KK context could be a good context to unite YM and GR. This is wildly different than the usually suggested non-abelian KK context with structural group G (Kerner 1968), (Witten 1981), (Bott 1985). Also, above the Higgs was related to the vector field v, not to an internal component of a gauge field as in e.g. (Manton 1979), (Panico & al. 2006). Also, remark that the idea of embedding the electroweak structural group $\mathrm{U}(1)\times\mathrm{SU}(2)$ inside the structural group of the tangent frame bundle $\mathrm{Fr}(Q)$ in a 5D KK context was already mentioned back in (Salam & Strathdee 1982), p.347.

Now, as a reminder, the original aim of this present document is not to unite either the Higgs with gauge fields nor unite gauge fields with gravity. The original aim is merely to see mass as a momentum. Doing so, we came upon gauge fields considerations because of the dilaton / Higgs analogy. Now, let's get back to mass as a momentum.

15. NEGATIVE MASSES, INVISIBLE FIFTH DIMENSION AND ALL THAT

One key feature of the *mass = momentum hypothesis* (116):

$$p_4 = h^{3/2}mc$$

is that mass can be negative. Suppose that an electron e^- and a positron e^+ are annihilated in two photons γ:

$$e^- + e^+ \rightsquigarrow \gamma + \gamma$$

In the right hand side, each photon γ has a vanishing mass so that the right hand side has a vanishing total p_4. Because of momentum conservation, the left hand side must also have a vanishing total p_4. Because the electron has a positive mass, it follows that the positron has a negative mass.

Denote by $m_- > 0$ the mass of the electron and $q_- < 0$ its electric charge. Denote by $m_+ = -m_- < 0$ the mass of the positron and $q_+ = -q_- > 0$ its electric charge.

According to (99), when $\tilde{\beta} = 0$ and $h = 1$, the equation of motion of a massive point particle parametrized by its proper time τ' in spacetime $(\tilde{Q}, \hat{g})$ is:

$$\frac{d^2\gamma^k}{d\tau'^2} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i}{d\tau'}\frac{d\gamma^j}{d\tau'} = 0 \tag{138}$$

In this equation, there is no EM Lorentz force. Adding it by hand to (138) we get the equation of motion of an electrically charged particle of mass m and electric charge q:

$$\frac{d^2\gamma^k}{d\tau'^2} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i}{d\tau'}\frac{d\gamma^j}{d\tau'} = \frac{q}{m}\hat{g}^{jk}F_{ij}\frac{d\gamma^i}{d\tau'} \tag{139}$$

Thus, the equations of motion of the electron and of the positron are respectively:

$$\frac{d^2\gamma^k_-}{d\tau'^2_-} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i_-}{d\tau'_-}\frac{d\gamma^j_-}{d\tau'_-} = \frac{q_-}{m_-}\hat{g}^{jk}F_{ij}\frac{d\gamma^i_-}{d\tau'_-} \tag{140}$$

$$\frac{d^2\gamma^k_+}{d\tau'^2_+} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i_+}{d\tau'_+}\frac{d\gamma^j_+}{d\tau'_+} = \frac{q_+}{m_+}\hat{g}^{jk}F_{ij}\frac{d\gamma^i_+}{d\tau'_+} \tag{141}$$

Assuming that:

$$\frac{d\tau'_-}{d\tau'_+} = -1 \tag{142}$$

we can reformulate (141) as:

$$\frac{d^2\gamma^k_+}{d\tau'^2_-} + \hat{\Gamma}^k_{ij}\frac{d\gamma^i_+}{d\tau'_-}\frac{d\gamma^j_+}{d\tau'_-} = -\frac{q_-}{m_-}\hat{g}^{jk}F_{ij}\frac{d\gamma^i_+}{d\tau'_-} \tag{143}$$

Thus, comparing (140) and (143), we see that the dynamic of the positron is the expected dynamic of a positron. Hence, there is no immediate problem with having a positron of negative mass. However, this is at the cost of assuming that the proper time of the electron and the proper time of the electron are related as (142). Nevertheless, this change of sign of the proper time τ' was already predicted by (98):

$$\frac{d\tau'}{d\tau} = \frac{p_4}{h^{1/2}m_{\mathrm{P}}c}$$

Changing the sign of p_4 amounts to change the sign of the evolution of the proper time τ'. Thus, changing the sign of mass m amounts to change the sign of the evolution of τ', as was done in (141).

When $\tilde{\beta} = 0$, (82) becomes:

$$p_4 = -h^2 m_{\mathrm{P}}\frac{dx^4}{d\tau} \tag{144}$$

Comparing (98) to (144) we get a relationship between proper time τ' and the x^4 coordinate:

$$c\,d\tau' = -h^{3/2}dx^4 \tag{145}$$

Thus, up to sign and scaling by h, the proper time τ' corresponds to the fifth dimension x^4 and mass is its corresponding momentum. Letting $ds = c\,d\tau'$, the relationship (145) implies:

$$ds = -h^{3/2}dx^4 \tag{146}$$

$$\partial_s = -h^{-3/2}\partial_4 \tag{147}$$

Thus, the mass operator (118) we found in §10:

$$\hat{m}c = -i\hbar h^{-3/2}\partial_4 \tag{148}$$

matches, up to a sign, the conjectured mass operator (3) of §1:

$$\hat{m}c = -i\hbar\partial_s$$

If one wants at all cost to match the signs of these two mass operators, then one would need to change either the sign in (79) either change the sign in (98). Rewriting all the above equations with such a different sign convention is left as an exercise.

The idea of defining a mass operator is not new and lies here and there within the KK literature, see e.g. (Witten 1981) p.420, (Salam & Strathdee 1982) p.350-351. However, the effective mass found in the KK literature is often an effective mass coming from the electric charge. Doing so, an electrically neutral particle has a vanishing mass unless a non-vanishing μ is considered in the 5D KG equation (73) as was done in e.g. (Souriau 1962). Above, mass was defined in terms of p_4, independently of any notion of electrical charge.

Remark that a genuine *mass squared operator* was defined in (Sniatycki 1980), p.174, via geometric quantization procedures. This is done in a 4D KG context without the need for a fifth dimension. In this mass squared operator, there is a scalar curvature term $R/6$. Such an added scalar curvature is related to an eventually added "divergence term" in the mass operator (118), as mentioned back in §10. So, the mass operator (118) should still be tweaked a little bit.

Now, there is a physical day to day issue with the existence of a fifth dimension, namely that we do not see such a dimension. Historically (Klein 1926), the fifth dimension was thought to be a tiny circle S^1 whose length is roughly the *Planck length* $\ell_{\text{P}} := \sqrt{\hbar G/c^3} \approx 10^{-35}$m. While this is the right length to get the EM interpretation of KK theory, it leads to masses being $\approx 10^{20}$ too heavy as described in §9. Hence, such a tiny circle is discarded. Instead, this fifth dimension could be a circle S^1 whose length lies between a millimeter and a meter. This the size considered in (Arkani-Hamed et al. 1998), (Dienes & al. 1998), (Appelquist et al. 2001).

Here is another possible motivation for such a size. Our Universe is filled with an omnipresent ambiant bath of photons called the *cosmic microwave background* (CMB). This background radiation has the spectrum of a cold radiating black body at $T_{\text{CMB}} \approx 2.725$K. Using Wien's displacement law $\lambda_{\text{CMB}} = b/T_{\text{CMB}}$, where $b \approx 2.898 \times 10^{-3}$ m $\cdot$ K is Wien's constant, the CMB's peak wavelength λ_{CMB} is roughly 1.063 mm long. The CMB is supposed to come from a distant past. Because of the equipartition theorem, the CMB should spread its energy equally in the (x, y, z) directions and in the s direction. Thus, the CMB's spectrum should be the same all over (x, y, z, s). If the CMB's spectrum is indeed related, via Fourier modes, to a finite length or thickness of the fifth dimension s, then this length should be scaled in, say, decimeters. Although such an eventual relationship between the CMB and the length of the fifth dimension is compatible with the standard electroweak / Higgs picture, it is incompatible with the MMH picture. Indeed, suppose that the CMB's spectrum is the same all over (x, y, z, s). Then, the CMB's photons will have a non-vanishing momentum p_4 and hence a non-vanishing mass, which is not true. Thus, the bad news is that according to the MMH, the CMB's spectrum has nothing to do with the length of the fifth dimension. But, the good news is that we now know why we do not see the fifth dimension.

In the MMH picture, photons have a vanishing p_4. Because they have a vanishing p_4, we do not *see* the fifth dimension. Moreover, back in §8, 4D timelike motion corresponds to 5D lightlike motion so that massive particles move at the speed of light in the 5D space, mostly in the direction of proper time $\tau' \propto x^4$. Thus, because matter is moving at the speed of light mostly towards the fifth dimension, there is no reason why we would *see* such a fifth dimension. Hence, it is not necessary to compactify the fifth dimension to a circle S^1 to hide it. We can keep it as being $\mathbb{R}$, as was done in §6.

In §8, 5D lightlike geodesics were the light rays of the WKB wave $\psi = e^{iS/\hbar}$ on (Q, g) satisfying the 5D wave equation $\pentagon\psi = 0$. Let's consider $(Q, g) = (\mathbb{R}^5, \eta)$. If a *signal* is emitted at the origin $(x^0, ..., x^4) = (0, ..., 0)$, its propagation corresponds to a *light cone* inside $\mathbb{R}^5$. At each time $t = x^0/c$, the signal propagates as a *wavefront* $S^3 \subset \mathbb{R}^4$ centered at $(x^1, ..., x^4) = (0, ..., 0)$ of radius $ct = x^0$. The points of the wavefront S^3 propagate along radial light rays in $\mathbb{R}^4$. Suppose that us lightlike bipeds lie on the wavefront S^3 expanding at the speed of light. Then, we would *see* an expanding S^3 that we could call the *cosmos*. Now, let's rewind time. The cosmos goes back to a single singular point, the *Big Bang*. But, there is not really a singularity because the WKB approximation cease to be valid at the caustic. In any case, such an expanding cosmic wavefront theory is not compatible with the KK cylindrical hypothesis. In particular, mass would not be a constant of motion. So we need another picture.

Let's consider again $(Q, g) = (\mathbb{R}^5, \eta)$. Suppose that we are propagating in the ∂_4 direction so that the cylindrical condition is satisfied. Recall from §1 that the main motivation for the MMH was the relationship between the 4D KG equation (1) and the 5D wave equation (2).

For these two equations to be equivalent, we had to suppose that ψ is an eigenfunction of the mass operator $\hat{m}$. This, in return, implies that ψ is monochromatic along the fifth dimension:

$$\psi(x^0, ..., x^4) = \psi(x^0, ..., x^3)e^{\pm imcx^4/\hbar}$$

This implies that ψ has an infinite extent in the fifth dimension. Thus, all matter has an infinite extent in the fifth dimension. This does not seem realistic. However, if we suppose instead that matter has a finite extent in the fifth dimension, things get worse. Because matter propagates in the direction ∂_4 and because photons are propagating along constant slices $x^4 = const.$, much of the emitted photons are lost behind the matter wavefront propagating in the fifth direction. These photons are lost, *forever*. This is not physical either. Thus, it seems, the fifth dimension cannot be $\mathbb{R}$ but should be a circle S^1.

Let's consider $(Q = \mathbb{R}^4 \times S^1, g)$. Suppose that we are propagating in the S^1 direction and that the cylindrical condition is satisfied along this circle. Now, something strange is happening. Because of the identification of 4D timelike geodesics with 5D lightlike geodesics, we can identify 4D timelike dust with 5D lightlike dust. In particular, a static galaxy (i.e. a static 4D timelike dust) corresponds to a 5D beam of lightlike dust propagating along the circle S^1. Two neighbouring roughly static galaxies correspond to two nearly parallel beams of lightlike dust. In a 4D setting, it was shown by (Bonnor 1969) that there is no gravitational attraction between two parallel beams of lightlike dust. The same argument applies in a 5D setting. Thus, there should be roughly no gravitational attraction between two neighbouring roughly static galaxies. This seemingly absence of gravity between neighbouring timelike galactic dust applies not only to galaxies, but to all timelike dust, provided that it is static enough so that the corresponding 5D beams are parallel enough. Gravity in a 5D setting is often surprising. For example, gravity seems to disappear in a 5D KK magnetic monopole context (Gross & Perry 1983). For other problems regarding 5D gravity see e.g. (Coquereaux & Esposito-Farese 1990).

As a last encouraging remark, note that in a 5D setting instead of a 4D setting, (Bonnor 1969)'s potential (5.1) $\propto \ln(r)$ solution to its Poisson equation (4.7) $(\partial_1^2 + \partial_2^2)f(r) = 0$ now becomes a more familiar Coulomb potential $\propto 1/r$ solution to the Poisson equation $(\partial_1^2 + \partial_2^2 + \partial_3^2)f(r) = 0$. Doing so, even if there is almost no gravitational acceleration between neighbouring galaxies, these galaxies should nevertheless bend light rays according to a Coulombian potential.

16. CONCLUSION

Because of the similarity between the 4D KG equation and the 5D wave equation, it was hypothesized that mass is a momentum in a fifth dimension. Doing so, a 4D timelike motion corresponds to a 5D lightlike motion. This was done in a KK setting because of two reasons. First, because there is already a lot of literature regarding KK theory. Second, because the cylindrical condition on the metric implies that p_4 is a constant of motion. Doing so, when the dilaton is constant, mass is also a constant of motion.

In the 5D KK setting, there is a Maxwellian field ω and a dilaton field h. In the present context, the Maxwellian field is not interpreted as electromagnetism but as a new force. These two fields, the Maxwellian and the dilaton, were suggested to be responsible for the rotational curve of galaxies. However, this is incompatible with the observation of dark matter filaments.

Despite a lacking potential on the dilaton, a notion of mass for the dilaton field was derived. This mass is proportional to the pointwise energy $\|\omega\|_{\hat{g}}$ of the Maxwellian field ω. However, this mass is not compatible with the MMH picture because it contradicts the cylindrical hypothesis.

A possible link between the dilaton and the Higgs field is mentioned here and there in the KK literature. Such a link was discussed not only at a scalar level but also at a vectorial level. Doing so, the electroweak gauge bundle was seen as a subbundle of the 5D tangent frame bundle. Despite the analogies between the dilaton and the Higgs, the mechanism by which particles get their masses differ in the MMH picture and in the Higgs picture.

The invisibility of the fifth dimension follows from the fact that photons are massless. The global topology of the fifth dimension should be a circle S^1. Although no precise length is suggested for this circle, it is quite clear that it is not as small as the Planck length suggested by the usual EM KK theory.

Assuming that mass is a momentum, it was shown that positrons should have a negative mass. This follows from the conservation of momentum in the fifth dimension. This negative mass does not contradict the dynamic of the positron when it comes to the Lorentz force.

According to the MMH, the energy-momentum tensor of a 4D timelike dust should correspond to the energy-momentum tensor of a 5D lightlike dust. Because of the absence of gravity between parallel Bonnor lightlike dust beams, it seems that there should be almost no gravity between roughly static neighbouring galaxies. Nevertheless, because light propagates perpendicularly to matter in the 5D space, these galaxies should bend light as observed by a Coulombian potential.

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