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FROM THE EINSTEIN FIELD EQUATION TO THE FLAT SCHRÖDINGER EQUATION
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ABSTRACT

Starting from a general relativity setting, the goal of this document is to reach the flat Schrödinger equation. At first, the spacetime metric g is minimally coupled to a real function S in the Hilbert-Einstein action integral. The Euler-Lagrange equations corresponding to variations in g and in S are equivalent to the curved electromagnetically charged Klein-Gordon equation of a WKB wave whose amplitude corresponds to a conformal transformation and whose mass is defined in terms of scalar curvature. To reach the flat Schrödinger equation while keeping a constant non-vanishing scalar curvature, it is suggested to hide the curvature inside the villi of some tessellation of space and then to define an averaged flat metric. However, no metric solution is found solving the Einstein field equation inside a single villus. The flat Schrödinger equation is not reached.

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1. INTRODUCTION

The theory of general relativity (GR) and the quantum world are incompatible. For example, in some quantum context, one might want to consider the flat Klein-Gordon (KG) equation:

$$\eta^{ij}\partial_i\partial_j\psi = -\left(\frac{mc}{\hbar}\right)^2\psi \tag{1}$$

However, GR tells us that this flat KG equation cannot be entirely true. On the left hand side one finds a flat Minkowski metric η. On the right hand side one finds mass. GR tells us that mass curves spacetime. So, the spacetime metric on the left hand side cannot be a Minkowski metric. Worse, even if $m = 0$, the energy-momentum tensor T_{ij} due to ψ might not be zero. Hence, even the flat wave equation:

$$\eta^{ij}\partial_i\partial_j\psi = 0 \tag{2}$$

is not entirely physical. For the same reason, the flat Schrödinger equation (FSE):

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\left(\partial_x^2+\partial_y^2+\partial_z^2\right)\psi + eV\psi \tag{3}$$

is problematic.

Generalizing the flat equations (1,2,3) to a curved spacetime (M, g) can be done in different ways. A first way is to substitute the flat d'Alembert operator $\eta^{ij}\partial_i\partial_j$ by a Laplace-Beltrami-d'Alembert operator $\Box$. Doing so, for example, the wave equation (2) would become:

$$\Box\psi = 0$$

A second way is to ask for conformal invariance (Penrose 1964), (Chernikov & Tagirov 1968) or to ask that the equations must follow from geometric quantization procedures (Sniatycki 1980). Doing so, one would get instead:

$$\left(\Box - \frac{1}{6}R\right)\psi = 0$$

However, the challenge here is different. Instead of going *from flat to curved*, the goal is to go *from curved to flat*. More precisely, starting from a GR setting, the goal is to reach the flat Schrödinger equation (3). Here is step by step how this is to be done:

1. The GR setting to start with is the Hilbert-Einstein action integral $S_{\mathrm{HE}}[g,S]=\int(\frac{1}{2\kappa}R+L)\Omega_g$ coupling the metric g to a real function S via the Lagrangian density $L=-\frac{6}{2\kappa\hbar^2}\|dS+eA\|_g^2$. Here, A is a fixed electromagnetic differential 1-form A.

2. The Euler-Lagrange equations are:
$$R_{ij}=-6\hbar^{-2}(\partial_i S+eA_i)(\partial_j S+eA_j) \qquad (4)$$
$$0=\mathrm{Div}(\nabla S+eA^\sharp) \qquad (5)$$

3. Tracing (4) with g^{ij} we get:
$$R=-6\hbar^{-2}\|\nabla S+eA^\sharp\|_g^2 \qquad (6)$$

4. For a WKB wave $\psi:=e^{iS/\hbar}$ with constant amplitude and for $\Box_A$ the electromagnetically charged Laplace-Beltrami-d'Alembert operator, we have these real and imaginary components:
$$\Re(\psi^{-1}\Box_A\psi)=-\hbar^{-2}\left\|\nabla S+eA^\sharp\right\|_g^2$$
$$\Im(\psi^{-1}\Box_A\psi)=\hbar^{-1}\mathrm{Div}\left(\nabla S+eA^\sharp\right)$$
Hence, the above two equations (5,6) are equivalent to an electromagnetically charged version of the conformally invariant massless wave equation:
$$\left(\Box_A-\frac{1}{6}R\right)\psi=0 \qquad (7)$$

5. Under a conformal transformation $\hat{g}=e^{2\varphi}g$ and a change of amplitude $\hat{\psi}=e^{-\varphi}\psi$, we have:
$$\left(\hat{\Box}_A-\frac{1}{6}\hat{R}\right)\hat{\psi}=e^{-3\varphi}\left(\Box_A-\frac{1}{6}R\right)\psi$$
Doing so, the equation (7) implies that the WKB wave $\hat{\psi}=\rho e^{iS/\hbar}$ with amplitude $\rho=e^{-\varphi}$ satisfies this wave equation:
$$\left(\hat{\Box}_A-\frac{1}{6}\hat{R}\right)\hat{\psi}=0 \qquad (8)$$

6. Let's define a constant $r_{\mathrm{C}}:=\hbar/(mc)$ and assume that the scalar curvature $\hat{R}$ is constant:
$$\hat{R}=-\frac{6}{r_{\mathrm{C}}^2} \qquad (9)$$
Then, the equations (8,9) put together imply that the WKB wave $\hat{\psi}$ satisfies the KG equation:
$$\hat{\Box}_A\hat{\psi}=-\left(\frac{mc}{\hbar}\right)^2\hat{\psi} \qquad (10)$$

Hence, from the above somewhat arbitrary GR setting we get the curved KG equation.

The last step to do is to go from the curved KG equation (10) to the FSE (3). A first approach would be to take $\hat{g}=\eta$ and a t-independent $A_i=(V/c,0,0,0)$ so that the non-relativistic approximation of the KG equation (10) is, up to an energy shift $E\mapsto E-mc^2$, the FSE (3):
$$i\hbar\partial_t\psi=mc^2\psi-\frac{\hbar^2}{2m}\left(\partial_x^2+\partial_y^2+\partial_z^2\right)\psi+eV\psi \qquad (11)$$

However, this method does not work for us because for $\hat{g}=\eta$ the scalar curvature $\hat{R}$ vanishes, which contradicts the *mass = curvature hypothesis* (9).

A second approach would be to "flatten" the spacetime metric $\hat{g}$ on $\mathbb{R}^4=\mathbb{R}\times\mathbb{R}^3$ by hiding its non-vanishing constant scalar curvature $\hat{R}$ inside the villi of a tessellation of $\mathbb{R}^3$. Such a spatial folding in and out of $\mathbb{R}^3$ with constant spacetime curvature on $\mathbb{R}^4$ should be possible because $\hat{g}$ is a pseudo-Riemannian metric, not a Riemannian one. Doing so, $\hat{g}$ and the tessellation would give rise to an averaged flat metric $\bar{g}$ on which the FSE would be built. Physically, the WKB wave $\hat{\psi}$ propagating on the villi would be periodically lensed by each villus. However, this second approach is drastically harder to execute than the first approach. Indeed, here we need to:

1. Tessellate $\mathbb{R}^3$, e.g. via cuboctahedrons.

2. Solve (4,5,9) inside $\mathbb{R}\times$ (a single villus).

3. Extend this solution to the whole villi.

4. Define an averaged flat spacetime metric $\bar{g}$ from $\hat{g}$.

5. Show that the non-relativistic approximation of (10) averages via $\bar{g}$ to the FSE (11).

In this document, only the second point will be approached, without success.

Outline: §2-3 are an adapted pot-pourri of standard definitions listed in (Aubin-Cadot 2019) regarding Riemannian geometry and general relativity. §4 gives various quantities corresponding to the above mentioned Lagrangian. §5 re-does the steps above to go from a GR setting to the KG equation (10), but with a slightly slower pace and some added comments. §6 is a first attempt to solve the Einstein equation via a 4D time-independent Kaluza-Klein-style metric, without success. §7 is a second attempt with a slightly different metric as the one from §6, without success. §8 studies an equation that appeared both in §6 and §7. §9 concludes and suggests an opening.

2. RIEMANNIAN GEOMETRY

Let's recall some standard definitions of Riemannian geometry (Landau & Lifschitz 1971). Let (Q, g) be an n-dimensional pseudo-Riemannian manifold. Consider local coordinates (x^i) on $U \subset Q$. Let $\partial_i := \partial/\partial x^i$ be defined as $dx^i(\partial_j) = \delta^i_j$ where δ^i_j equals 1 if $i = j$ and 0 if $i \neq j$. The metric g is locally written:

$$g|_U = g_{ij} dx^i \otimes dx^j \tag{12}$$

where $g_{ij} := g(\partial_i, \partial_j)$. Repeated indices are summed over their range. Let g^{ij} be the coefficients of the inverse matrix $[g^{ij}] := [g_{ij}]^{-1}$, i.e. $\delta^i_j = g^{ik} g_{kj}$. The *Christoffel symbols*, the *Riemann curvature tensor*, the *Ricci curvature tensor*, the *scalar curvature* and the *Einstein tensor* of the metric g are respectively given by:

$$\Gamma^k_{ij} := (1/2) g^{km} (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}) \tag{13}$$
$$R^l_{kij} := \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik} \tag{14}$$
$$R_{ij} := R^k_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^l_{ij} \Gamma^k_{kl} - \Gamma^l_{ik} \Gamma^k_{jl} \tag{15}$$
$$R := g^{ij} R_{ij} \tag{16}$$
$$G_{ij} := R_{ij} - (1/2) R g_{ij} \tag{17}$$

The *Riemannian musicality isomorphisms* are:

$$\begin{aligned} \flat : TQ &\to T^*Q \\ v = v^i \partial_i &\mapsto v^\flat := g(v, \cdot) = g_{ij} v^i dx^j \end{aligned}$$

$$\begin{aligned} \sharp := \flat^{-1} : T^*Q &\to TQ \\ \alpha = \alpha_i dx^i &\mapsto \alpha^\sharp := g^{ij} \alpha_i \partial_j \end{aligned}$$

Let:

$$\begin{aligned} \|v\|^2_g &:= g(v, v) = g_{ij} v^i v^j \\ \|\alpha\|^2_g &:= g(\alpha^\sharp, \alpha^\sharp) = g^{ij} \alpha_i \alpha_j \end{aligned}$$

When a vector v and a 1-form α are mutually musical, we have $\|v\|_g = \|\alpha\|_g$. Let:

$$\begin{aligned} \det[g] &:= \det[g_{ij}] \\ |g| &:= |\det[g]| \\ d^n x &:= dx^1 \wedge .. \wedge dx^n \\ \Omega_g &:= |g|^{1/2} d^n x \end{aligned}$$

The *Levi-Civita covariant derivative* $\nabla_i := \nabla_{\partial_i}$ of tensors on Q is given in local coordinates by the Leibniz product rule and by:

$$\nabla_i f := \partial_i f \qquad \nabla_i(\partial_j) := \Gamma^k_{ij} \partial_k \qquad \nabla_i(dx^k) := -\Gamma^k_{ij} dx^j$$

On functions, one should not confuse the *gradient vector field of a function*:

$$\begin{aligned} \nabla f &:= (df)^\sharp \\ &= g^{ij} (\partial^i f) \partial_j \end{aligned}$$

and the *covariant derivative of a function*:

$$\begin{aligned} \nabla f &= df \\ &= (\partial_i f) dx^i \end{aligned}$$

These two ∇ are not the same. For this reason, on tensors ∇ will denote a covariant derivative but on functions ∇ will denote the gradient. Also, as an abuse of notation, the covariant derivative ∇T of a tensor T will often be denoted $\nabla T^{i_1 \dots i_p}_{j_1 \dots j_q}$. For example:

$$\nabla_i F_{jk} = \partial_i F_{jk} - \Gamma^l_{ij} F_{lk} - \Gamma^l_{ik} F_{jl}$$

The *covariant Hessian* of a function is:

$$\begin{aligned} H_{ij}(f) &:= (\nabla df)(\partial_i, \partial_j) \\ &= \left(\partial_i \partial_j - \Gamma^k_{ij} \partial_k\right) f \end{aligned}$$

The *covariant divergence* of a vector field $v = v^i \partial_i$ is defined implicitly as:

$$\mathrm{Div}(v) \cdot \Omega_g = \mathcal{L}_v \Omega_g$$

The covariant divergence is given explicitly as:

$$\begin{aligned} \mathrm{Div}(v) &= \partial_i v^i + v^i \Gamma^k_{ik} \\ &= |g|^{-1/2} \partial_i \left(|g|^{1/2} v^i\right) \end{aligned}$$

In particular, we have:

$$\mathrm{Div}(\partial_i) = \Gamma^k_{ik} = |g|^{-1/2} \partial_i \left(|g|^{1/2}\right)$$

For α a differential 1-form we have:

$$\mathrm{Div}(\alpha^\sharp) = g^{ij} (\partial_i \alpha_j - \Gamma^k_{ij} \alpha_k)$$

The *Laplace-Beltrami operator* acting on functions is:

$$\begin{aligned} \Delta f &:= \mathrm{Div}(\nabla f) \\ &= g^{ij} \left(\partial_i \partial_j - \Gamma^k_{ij} \partial_k\right) f \\ &= g^{ij} H_{ij}(f) \end{aligned}$$

On functions, the Laplace-Beltrami operator equals minus the Laplace-de Rham operator, i.e. $\Delta_{\mathrm{LB}} f = -\Delta_{\mathrm{LdR}} f$ where $\Delta_{\mathrm{LdR}} f := \delta df$ where δ is the Hodge codifferential. Here Δ will always denote Δ_{LB}.

In a $(+,-,-,-)$ pseudo-Riemannian setting, Δ is denoted by the *d'Alembertian operator* $\Box$. In such a

$(+,-,-,-)$ setting, for $A = A_i dx^i$ a real differential 1-form and for e and $\hbar$ some physical constants, let:

$$\begin{aligned} D_j &:= \partial_j + i\frac{e}{\hbar}A_j \\ \Box_A f &:= g^{ij}(D_iD_j - \Gamma^k_{ij}D_k)f \\ &= |g|^{-1/2}D_i\left(|g|^{-1/2}g^{ij}D_jf\right) \end{aligned}$$

In a $(+,-,-,-,-)$ pseudo-Riemannian setting, Δ is denoted by the *Souriau operator* $\pentagon$ and $\pentagon_A$ can be similarly defined.

Let $\gamma(\lambda)$ be a parametrized curve in (Q,g) and denote by a dot the derivative $d/d\lambda$. γ is said to be a *geodesic* of (Q,g) if it satisfies the *geodesic equation* $\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0$. γ is said to be an *integral curve* of a vector field v if it satisfies the *integral curve equation* $\dot\gamma = v\circ\gamma$. γ is said to be a *gradient curve* of a function f if it is an integral curve of the gradient vector field ∇f, i.e. $\dot\gamma = (\nabla f)\circ\gamma$. A vector field $v\in\mathfrak{X}(Q)$ is a *Killing vector field* of g if $\mathcal{L}_v g = 0$.

Under a *conformal rescaling* $\hat g = e^{2\varphi}g$ we have:

$$\begin{aligned} \hat g_{ij} &= e^{2\varphi}g_{ij} \\ \hat g^{ij} &= e^{-2\varphi}g^{ij} \\ \hat\Gamma^k_{ij} &= \Gamma^k_{ij} + \delta^k_i\partial_j\varphi + \delta^k_j\partial_i\varphi - g_{ij}\nabla^k\varphi \\ \hat R_{ij} &= R_{ij} - (n-2)\left(H_{ij}(\varphi) - (\partial_i\varphi)(\partial_j\varphi)\right) \\ &\quad - \left(\Delta\varphi + (n-2)\|d\varphi\|^2_g\right)g_{ij} \\ \hat R &= e^{-2\varphi}[R - 2(n-1)\Delta\varphi \\ &\quad -(n-2)(n-1)\|d\varphi\|^2_g] \\ |\hat g|^{1/2} &= e^{n\varphi}|g|^{1/2} \\ \Omega_{\hat g} &= e^{n\varphi}\Omega_g \\ \hat R\Omega_{\hat g} &= e^{(n-2)\varphi}[R - 2(n-1)\Delta\varphi \\ &\quad -(n-2)(n-1)\|d\varphi\|^2_g]\Omega_g \\ \alpha^{\hat\sharp} &= e^{-2\varphi}\alpha^\sharp \\ v^{\hat\flat} &= e^{2\varphi}v^\flat \\ \|v\|^2_{\hat g} &= e^{2\varphi}\|v\|^2_g \\ \|\alpha\|^2_{\hat g} &= e^{-2\varphi}\|\alpha\|^2_g \\ \widehat{\mathrm{Div}}(v) &= \mathrm{Div}(v) + nd\varphi(v) \\ \widehat{\mathrm{Div}}(\alpha^{\hat\sharp}) &= e^{-2\varphi}(\mathrm{Div}(\alpha^\sharp) + (n-2)\alpha(\nabla\varphi)) \\ \hat\nabla f &= e^{-2\varphi}\nabla f \\ \hat\Delta f &= e^{-2\varphi}(\Delta f + (n-2)(df)(\nabla\varphi)) \end{aligned}$$

In a $(+,-,-,-)$ pseudo-Riemannian setting, we have:

$$\begin{aligned} \left(\hat\Box - \frac{1}{6}\hat R\right)(f) &= e^{-3\varphi}\left(\Box - \frac{1}{6}R\right)(e^\varphi f) \\ \left(\hat\Box_A - \frac{1}{6}\hat R\right)(f) &= e^{-3\varphi}\left(\Box_A - \frac{1}{6}R\right)(e^\varphi f) \end{aligned}$$

3. GENERAL RELATIVITY

Let's recall some standard definitions of general relativity (Landau & Lifschitz 1971). Let (Q,g) be a 4-dimensional $(+,-,-,-)$ pseudo-Riemannian manifold called *spacetime*. The *Einstein constant* is defined as $\kappa := 8\pi G/c^4$. Let L be some Lagrangian density. The *Hilbert energy-momentum tensor* of L is defined as:

$$T_{ij} := 2\frac{1}{|g|^{1/2}}\frac{\partial}{\partial g^{ij}}(L|g|^{1/2}) \tag{18}$$

$$= 2\frac{\partial L}{\partial g^{ij}} + g_{ij}L \tag{19}$$

The *Laue scalar* of L is:

$$T := g^{ij}T_{ij} \tag{20}$$

The *Hilbert-Einstein action integral* S_{HE} is defined as:

$$S_{\mathrm{HE}}[g,S] := \int_Q \left(\frac{1}{2\kappa}R + L\right)\Omega_g \tag{21}$$

Using the identity:

$$\begin{aligned} G_{ij} &= \frac{1}{|g|^{1/2}}\frac{\delta}{\delta g^{ij}}(R|g|^{1/2}) \\ &= R_{ij} - \frac{1}{2}g_{ij}R \end{aligned}$$

the Euler-Lagrange equation according to variations in g_{ij} of the HE action integral gives the *Einstein field equation (EFE) of general relativity*:

$$G_{ij} = \kappa T_{ij} \tag{22}$$

Tracing both sides of the Einstein equation (22) with g^{ij} and using the fact that $g^{ij}G_{ij} = -R$, one finds:

$$R = -\kappa T \tag{23}$$

Let's define this tensor:

$$K_{ij} := T_{ij} - \frac{1}{2}g_{ij}T \tag{24}$$

Using (23), the Einstein equation (22) is equivalent to the *trace-reversed EFE*:

$$R_{ij} = \kappa K_{ij} \tag{25}$$

Assume that the Lagrangian L is defined over real functions $S : Q \to \mathbb{R}$. Then, the Euler-Lagrange equation according to variations in S of (21) gives this other field equations:

$$\partial_i\left(\frac{\partial L}{\partial(\partial_i S)}|g|^{1/2}\right) = \frac{\partial L}{\partial S}|g|^{1/2} \tag{26}$$

4. A SPECIFIC LAGRANGIAN

Let's fix an electric charge e and a real background electromagnetic differential 1-form $A \in \Omega^1(Q;\mathbb{R})$. Let's choose a specific Lagrangian density L defined over real functions S on Q:

$$L = -\frac{6}{2\kappa\hbar^2}\|dS + eA\|_g^2 \tag{27}$$

Here, S has the physical units of $\hbar$ and the term $1/\kappa$ is only here to match physical units. To simplify the notation, let's define respectively a differential 1-form and a vector field over Q:

$$\alpha := dS + eA \in \Omega^1(Q) \tag{28}$$
$$v := \nabla S + eA^\sharp \in \mathfrak{X}(Q) \tag{29}$$

They satisfy $\alpha = v^\flat$ and $v = \alpha^\sharp$. Doing so, the Lagrangian density L can be written equivalently as:

$$L = -\frac{6}{2\kappa\hbar^2}\|\alpha\|_g^2 = -\frac{6}{2\kappa\hbar^2}\|v\|_g^2 \tag{30}$$

Using (19) and (30), we get:

$$T_{ij} = -\frac{6}{\kappa\hbar^2}\left(\alpha_i\alpha_j - \frac{1}{2}g_{ij}\|\alpha\|_g^2\right) \tag{31}$$
$$T^{ij} = -\frac{6}{\kappa\hbar^2}\left(v^i v^j - \frac{1}{2}g^{ij}\|v\|_g^2\right) \tag{32}$$

The Laue scalar becomes:

$$T = \frac{6}{\kappa\hbar^2}\|\alpha\|_g^2 = \frac{6}{\kappa\hbar^2}\|v\|_g^2 \tag{33}$$

The K_{ij} tensor becomes:

$$K_{ij} = -\frac{6}{\kappa\hbar^2}\alpha_i\alpha_j \tag{34}$$
$$K^{ij} = -\frac{6}{\kappa\hbar^2}v^i v^j \tag{35}$$

The trace-reversed EFE (25) becomes:

$$R_{ij} = -\frac{6}{\hbar^2}\alpha_i\alpha_j \tag{36}$$
$$R^{ij} = -\frac{6}{\hbar^2}v^i v^j \tag{37}$$

Equation (23) becomes:

$$R = -\frac{6}{\hbar^2}\|\alpha\|_g^2 = -\frac{6}{\hbar^2}\|v\|_g^2 \tag{38}$$

The Euler-Lagrange equation (26) following from variations in S is the divergence-free equation:

$$\mathrm{Div}(v) = 0 \tag{39}$$

When $A = 0$, this Euler-Lagrange equation is equivalent to the wave equation $\Box S = 0$.

5. FROM GR TO THE KG EQUATION

For our specific Lagrangian L, the field equations (38,39) can be written as:

$$-\frac{1}{6}\hbar^2 R = \|\nabla S + eA^\sharp\|_g^2 \tag{40}$$
$$0 = \mathrm{Div}(\nabla S + eA^\sharp) \tag{41}$$

It happens that these two equations are equivalent to an electromagnetically charged version of the conformally invariant wave equation (Penrose 1964), (Chernikov & Tagirov 1968), (Sniatycki 1980):

$$\left(\Box_A - \frac{1}{6}R\right)\psi = 0 \tag{42}$$

Here, ψ is a WKB wave with constant amplitude:

$$\psi = e^{iS/\hbar}$$

and $\Box_A$ is the usual electromagnetically charged Laplace-Beltrami-d'Alembert operator as defined in §2. The equivalence of (40,41) with (42) follows from the fact that we have these real and imaginary components:

$$\Re(\psi^{-1}\Box_A\psi) = -\hbar^{-2}\left\|\nabla S + eA^\sharp\right\|_g^2$$
$$\Im(\psi^{-1}\Box_A\psi) = \hbar^{-1}\mathrm{Div}\left(\nabla S + eA^\sharp\right)$$

The wave equation (42) is conformally invariant in that, as mentioned back in §2, for a conformal transformation $\hat{g} = e^{2\varphi}g$ it satisfies:

$$\left(\hat{\Box}_A - \frac{1}{6}\hat{R}\right)(e^{-\varphi}\psi) = e^{-3\varphi}\left(\Box_A - \frac{1}{6}R\right)(\psi) \tag{43}$$

Although proving (43) is left as an exercise for the conscientious reader, a clean way to show it is to use this identity:

$$\Box_A\psi = \Box\psi + \frac{2ie}{\hbar}A(\nabla\psi) - \psi\frac{e^2}{\hbar^2}\|A\|_g^2 + \psi\frac{ie}{\hbar}\mathrm{Div}(A^\sharp)$$

We want to reach the FSE. In the Schrödinger equation, the wave function does not have a constant amplitude. However, the WKB wave $\psi = e^{iS/\hbar}$ has a constant amplitude. We need to give ψ an amplitude function. For this, two choices are possible. A first choice would be to add an amplitude function ρ inside the Lagrangian (27). However, in this case:

1. (31,33,34) would get be more complicated.
2. (36) would become complicated.
3. There would be a new Euler-Lagrange equation corresponding to variations in ρ.

4. It would be easier to define L directly in terms of a wave ψ and not in terms of S and ρ separately. But we would loose the interesting relationship between the EFE and the wave equation (42) via the WKB wave.

Let's give ψ an amplitude ρ in a different way. Taking a look at the equation (43), we see that in the left hand side, $e^{-\varphi}$ seems to modify the amplitude. Thus, the second choice to give ψ an amplitude is to define an amplitude via a conformal transformation:

$$\hat{g} := e^{2\varphi} g \tag{44}$$
$$\rho := e^{-\varphi} \tag{45}$$
$$\hat{\psi} := \rho\psi = \rho e^{iS/\hbar} \tag{46}$$

Doing so, because of (42) and (43), we know that the WKB wave with amplitude $\hat{\psi}$ must satisfy this electromagnetically charged wave equation:

$$\left(\hat{\Box}_A - \frac{1}{6}\hat{R}\right)\hat{\psi} = 0 \tag{47}$$

Now that the WKB wave has an amplitude function, we still lack the mass term m that lies in the FSE. In the wave equation (47) there is no mass term m. To get a mass term m, two choices are possible. A first usual but boring choice would be to add a mass term m in the Lagrangian (27). However, in this case (31,33,34) would get more complicated so that (36) would also get more complicated. Let's not do that. Instead, remark that the electromagnetically charged KG equation:

$$\hat{\Box}_A\hat{\psi} = -\left(\frac{mc}{\hbar}\right)^2\hat{\psi} \tag{48}$$

looks a lot like (47). Let's hypothesize that *they are the same equation*. This amounts to hypothesize that the scalar curvature $\hat{R}$ is constant and equal to:

$$\hat{R} = -6\left(\frac{mc}{\hbar}\right)^2 \tag{49}$$

Let's call this the *mass = curvature hypothesis*. Let's define the *Compton radius* as:

$$r_{\mathrm{C}} := \frac{\hbar}{mc}$$

It is related to the *Compton wavelength* λ_{C} in that λ_{C} can be seen as the circumference of a circle of radius r_{C}:

$$\lambda_{\mathrm{C}} = 2\pi r_{\mathrm{C}}$$

Now, the hypothesis (49) can be written concisely as:

$$\hat{R} = -\frac{6}{r_{\mathrm{C}}^2} \tag{50}$$

Doing so, the WKB wave with amplitude $\hat{\psi} = \rho e^{iS}$ satisfies the electromagnetically charged KG equation (48).

The hypothesis (50) combined with the conformal transformation (44) implies a relationship between the scalar curvature R and the Compton radius r_{C}. To simplify subsequent calculations, the amplitude function ρ will be supposed constant. The function P is then also supposed constant. This is the coarsest WKB approximation possible, namely that the WKB wave $\hat{\psi}$ has a constant amplitude. A possible justification for such an approximation will come in a later remark. The relationship between R and r_{C} is then:

$$R = \frac{\hat{R}}{P} = -\frac{6}{r_{\mathrm{C}}^2 P} = -\frac{6}{r_{\mathrm{C}}^2\rho^2} \tag{51}$$

Hence, from now on, we only need to work with the metric g and not $\hat{g}$.

Remark: The equation (38), the definitions (44,45) and the hypothesis (49) put together imply:

$$\frac{m^2c^2}{\rho^2} = \|\alpha\|_g^2 = \|v\|_g^2 \tag{52}$$

This can be reformulated as:

$$m^2c^2 = \|\alpha\|_{\hat{g}}^2 \tag{53}$$

This equation is usually an *ad hoc* condition. Here, however, this equation follows from more fundamental first principles. This is a direct consequence of forging the effective mass out of geometry.

Remark: The hypothesis (49) relating mass m to the scalar curvature $\hat{R}$ is part of a bigger project to define the effective mass in geometrical terms. Another approach would be, for example, to define the effective mass as a momentum in a penta-dimensional setting (Aubin-Cadot 2018), (Aubin-Cadot 2019).

Remark: The physicality of using a scalar field in GR is greatly discussed in (Bercelo & Visser 2002). They list various physical scalar fields e.g. scalar mesons, Higgs boson, axion, inflaton, Brans-Dicke scalar. However, they forget to list one "physical" scalar "field", namely the phase function S of a WKB wave.

Remark: A conformally invariant wave equation such as $(\Box - R/6)f = 0$ is usually found as being the Euler-Lagrange equation of a modified Hilbert-Einstein action where there is a notion of *conformal coupling*, see e.g. (Brown & Ottewill 1983), (Madsen 1988). Despite the fact that we did find a conformally invariant wave equation (47), no such conformal coupling was supposed in (21), only a *minimal coupling*.

Remark: Let's define a *light ray* of ψ as a gradient curve of the phase function S (see e.g. (Misner

& Thorne & Wheeler 1973), p.573). When $A = 0$, the equations (40,41,51) imply that the *light rays* of ψ are geodesics in (M, g). Because ρ is assumed constant, these geodesics of (M, g) correspond via a reparametrization to geodesics of $(M, \hat{g})$.

Remark: It could be a good idea to add $-\frac{1}{4\mu_0}\|F\|_g^2$ in the Lagrangian density (27) for $F := dA$ the electromagnetic Maxwell tensor. In such a scenario, (31) and (34) would be augmented by some electromagnetic contribution. The Euler-Lagrange equation corresponding to variations in A is $\delta F \propto \alpha$ and, combined with $dF = d^2A = 0$, gives Maxwell's equations. The Laue scalar due to F vanishes. Hence, F would not alter the effective mass of the WKB wave.

Remark: Assume again that the amplitude $\rho = e^{-\varphi}$ is constant (or varies slowly enough) in the vicinity of a villus compared to $\hat{g}$ so that the derivatives of φ are discarded in the formula relating R and $\hat{R}$. Doing so, we have $\hat{R} = PR$ as in (51). Hence, $\hat{R}$ is proportional to R. Assume moreover that the 4D spacetime scalar curvature R of g is proportional to the 3D spatial scalar curvature. Then, it is plausible that the 3D spatial scalar curvature of a villus be somewhat close to $\pm 6/r_{\mathrm{C}}^2$. It happens that $6/r_{\mathrm{C}}^2$ is the scalar curvature of a standard 3-sphere S^3 of radius r_{C}. Comparing the Compton radius r_{C} of an electron of mass $m = m_e$ to the size of a hydrogen atom being described by the Bohr radius $a_0 = r_{\mathrm{C}}/\alpha$, we get:

$$\frac{r_{\mathrm{C}}}{a_0} = \alpha \approx \frac{1}{137}$$

Here α is not the differential 1-form (28) but a physical constant without physical units known as the *fine structure constant*. Hence, the size of a villus should be close to $1/137$ the size of a hydrogen atom. The WKB approximation that P varies slowly enough at the scale of a villus then seems justified.

Remark: When ρ and $\hat{g}$ are t-independent there seems to be a link between the Hilbert-Einstein integral over a long thin cylinder $U = [t_0, t_1] \times B^3 \subset \mathbb{R} \times \mathbb{R}^3$ and the probability to find the particle in the small ball B^3:

$$\int_U R\Omega_g = \int_U e^{-2\varphi}\hat{R}\Omega_{\hat{g}} = -\frac{6}{r_{\mathrm{C}}^2}\int_U \hat{\psi}^\dagger\hat{\psi}\Omega_{\hat{g}} \propto \int_{B^3}\hat{\psi}^\dagger\hat{\psi}\Omega_{\tilde{g}}$$

Here, $\Omega_{\tilde{g}}$ is some volume 3-form on the space $\mathbb{R}^3$ induced by $\hat{g}$ on $\mathbb{R}^4$.

Remark: Because of the villi, the periodically lensed light rays of the WKB wave would draw paths that could look like a random walk. Thus, maybe the path integral formalism could help going from the curved KG equation to the FSE.

Now, we need to split time and space.

6. Splitting time and space

We want to reach the FSE. In this equation, time and space are *splitted*, meaning that spacetime is a Cartesian product of time and space:

$$Q = \mathbb{R} \times \tilde{Q}$$

This indicates that we need to consider a spacetime metric g where notions of time and space are explicit. The most generic such metric on $Q = \mathbb{R} \times \tilde{Q}$ is of this kind:

$$g = h^2\beta \otimes \beta - \tilde{g} \tag{54}$$

Here lies a function $h : Q \to \mathbb{R}_+$, a differential 1-form $\beta = cdt + \tilde{\beta} \in \Omega^1(Q; \mathbb{R})$ for $\tilde{\beta} = \sum_{i\neq 0}\tilde{\beta}_i dx^i$ a differential 1-form without dt and $\tilde{g}$ a $(+,+,+)$ Riemannian metric without dt on the 3D space $\tilde{Q}$. For simplicity, assume that g is time-independent so that h, β, $\tilde{\beta}$ and $\tilde{g}$ are all time-independent. The indices of $\tilde{\beta}_i$ and $\tilde{g}_{ij}$ run over $i = 1, 2, 3$.

The metric (54) has the same shape and properties of a *Kaluza-Klein* (KK) metric. However, it is the time components of a $(+,-,-,-) = (+) - (+,+,+)$ spacetime metric g that is being singled out, not the *fifth component* 5D metric. The t-independence of g corresponds to the so-called *cylindrical condition* where the KK metric is independent of the fifth coordinate. The advantage to use a KK-ish metric is that it is quite generic and that there is already a broad literature on KK theory at our disposal.

The t-independence of g can be concisely written as $\mathcal{L}_{\partial_0} g = 0$. Explicitly, we have:

$$\begin{aligned}
\partial_0 h &= 0\\
\tilde{\beta}(\partial_0) &= 0\\
\partial_0 \tilde{\beta}_i &= 0\\
\tilde{g}(\partial_0, \cdot) &= \tilde{g}(\cdot, \partial_0) = 0\\
\partial_0 \tilde{g}_{ij} &= 0
\end{aligned}$$

where $\tilde{\beta}^i := \tilde{g}^{ij}\tilde{\beta}_j$. Let $\|\tilde{\beta}\|_{\tilde{g}}^2 := \tilde{g}^{ij}\tilde{\beta}_i\tilde{\beta}_j$. The components g_{ij} of the metric (54) and its inverse matrix g^{ij} are explicitly given by:

$$g_{ij} = \begin{cases} h^2 & \text{for } i = 0,\ j = 0\\ h^2\tilde{\beta}_j & \text{for } i = 0,\ j \neq 0\\ h^2\tilde{\beta}_i & \text{for } i \neq 0,\ j = 0\\ h^2\tilde{\beta}_i\tilde{\beta}_j - \tilde{g}_{ij} & \text{for } i \neq 0,\ j \neq 0 \end{cases}$$

$$g^{ij} = \begin{cases} h^{-2} - \|\tilde{\beta}\|_{\tilde{g}}^2 & \text{for } i = 0,\ j = 0\\ \tilde{\beta}^j & \text{for } i = 0,\ j \neq 0\\ \tilde{\beta}^i & \text{for } i \neq 0,\ j = 0\\ -\tilde{g}^{ij} & \text{for } i \neq 0,\ j \neq 0 \end{cases}$$

A direct calculation shows that $\delta^i_j = g^{ik} g_{kj}$. The following explicit formulas for Γ^k_{ij}, for R_{ij} and R are deduced in a $(+,-,-,-) = (+) - (+,+,+)$ context from the $(+,-,-,-,+) = (+,-,-,-) + (+)$ context computed by (Williams 2015). For $i,j,k,l \neq 0$:

$$\begin{aligned}
\omega &:= \omega_{ij} dx^i \otimes dx^j = (\partial_i \tilde{\beta}_j - \partial_j \tilde{\beta}_i) dx^i \otimes dx^j \\
\|\omega\|^2_{\tilde{g}} &:= \tilde{g}^{ik} \tilde{g}^{jl} \omega_{ij} \omega_{kl} \\
\tilde{\nabla}_i \omega_{jk} &= \partial_i \omega_{jk} - \tilde{\Gamma}^l_{ij} \omega_{lk} - \tilde{\Gamma}^l_{ik} \omega_{jl}
\end{aligned}$$

For $i,j,k,l \neq 0$, the Christoffel symbols Γ^k_{ij} of g are:

$$\begin{aligned}
\Gamma^0_{00} &= -\tilde{\beta}^l h \partial_l h \\
\Gamma^0_{i0} &= \Gamma^0_{0i} = -\frac{1}{2} \tilde{\beta}^l h^2 \omega_{li} - \tilde{\beta}^l \tilde{\beta}_i h \partial_l h + \partial_i \ln h \\
\Gamma^0_{ij} &= -\tilde{\beta}_l \tilde{\Gamma}^l_{ij} - \frac{1}{2} \tilde{\beta}^l \tilde{\beta}_j h^2 \omega_{li} - \frac{1}{2} \tilde{\beta}_i \tilde{\beta}^l h^2 \omega_{lj} \\
&\quad + \frac{1}{2} (\partial_i \tilde{\beta}_j + \partial_j \tilde{\beta}_i) + \tilde{\beta}_j \partial_i \ln h + \tilde{\beta}_i \partial_j \ln h \\
&\quad - \tilde{\beta}_i \tilde{\beta}^l \tilde{\beta}_j h \partial_l h \\
\Gamma^k_{00} &= \tilde{g}^{kl} h \partial_l h \\
\Gamma^k_{i0} &= \Gamma^k_{0i} = -\frac{1}{2} \tilde{g}^{kl} (h^2 \omega_{il} - 2 \tilde{\beta}_i h \partial_l h) \\
\Gamma^k_{ij} &= \tilde{\Gamma}^k_{ij} - \frac{1}{2} \tilde{g}^{kl} (\tilde{\beta}_j h^2 \omega_{il} + \tilde{\beta}_i h^2 \omega_{jl} - 2 \tilde{\beta}_i \tilde{\beta}_j h \partial_l h)
\end{aligned}$$

The Ricci and scalar curvature of g are given for $i,j,k,l \neq 0$ as follow:

$$\begin{aligned}
R_{00} &= h \tilde{\Delta} h + \frac{1}{4} h^4 \|\omega\|^2_{\tilde{g}} \\
R_{i0} &= -\frac{1}{2} h^2 \tilde{g}^{jk} \tilde{\nabla}_j \omega_{ik} - \frac{3}{2} \tilde{g}^{jk} h (\partial_k h) \omega_{ij} + \tilde{\beta}_i R_{00} \\
R_{ij} &= \tilde{R}_{ij} + \frac{1}{2} h^2 \tilde{g}^{kl} \omega_{ik} \omega_{jl} - h^{-1} \tilde{H}_{ij}(h) \\
&\quad - \tilde{\beta}_i \tilde{\beta}_j R_{00} + \tilde{\beta}_i R_{j0} + \tilde{\beta}_j R_{i0} \\
R &= -\tilde{R} + 2h^{-1} \tilde{\Delta} h - \frac{1}{4} h^2 \|\omega\|^2_{\tilde{g}}
\end{aligned}$$

Now, one could easily get lost in this happy carnival of notation. Where to go? Suppose that $\omega = 0$, i.e. suppose that $\tilde{\beta}$ is closed. Then, the Ricci components and the scalar curvature become:

$$\begin{aligned}
R_{00} &= h \tilde{\Delta} h \\
R_{i0} &= \tilde{\beta}_i h \tilde{\Delta} h \\
R_{ij} &= \tilde{\beta}_i \tilde{\beta}_j h \tilde{\Delta} h + \tilde{R}_{ij} - h^{-1} \tilde{H}_{ij}(h) \\
R &= -\tilde{R} + 2h^{-1} \tilde{\Delta} h
\end{aligned}$$

Suppose moreover that for $i,j \neq 0$ we have:

$$\tilde{R}_{ij} = h^{-1} \tilde{H}_{ij}(h) \tag{55}$$

Tracing both sides of (55) by $\tilde{g}^{ij}$ gives:

$$\tilde{R} = h^{-1} \tilde{\Delta} h \tag{56}$$

The Ricci components and the scalar curvature of the spacetime metric (54) are then given for $i,j \neq 0$ by:

$$R_{00} = h \tilde{\Delta} h \tag{57}$$

$$R_{i0} = \tilde{\beta}_i h \tilde{\Delta} h \tag{58}$$

$$R_{ij} = \tilde{\beta}_i \tilde{\beta}_j h \tilde{\Delta} h \tag{59}$$

$$R = \tilde{R} = h^{-1} \tilde{\Delta} h \tag{60}$$

We want to solve the EFE (36) and the hypothesis (51):

$$R_{ij} = -\frac{6}{\hbar^2} \alpha_i \alpha_j \tag{61}$$

$$R = -\frac{6}{r_{\mathrm{C}}^2 \rho^2} \tag{62}$$

That is, for $i,j \neq 0$ we want to solve:

$$-\frac{6}{\hbar^2} \alpha_0 \alpha_0 = h \tilde{\Delta} h \tag{63}$$

$$-\frac{6}{\hbar^2} \alpha_0 \alpha_i = \tilde{\beta}_i h \tilde{\Delta} h \tag{64}$$

$$-\frac{6}{\hbar^2} \alpha_i \alpha_j = \tilde{\beta}_i \tilde{\beta}_j h \tilde{\Delta} h \tag{65}$$

$$-\frac{6}{r_{\mathrm{C}}^2 \rho^2} = R = \tilde{R} = h^{-1} \tilde{\Delta} h \tag{66}$$

Doing so, for $i,j \neq 0$ we get:

$$\begin{aligned}
\alpha_0 \alpha_0 &= (hmc\rho^{-1})(hmc\rho^{-1}) \\
\alpha_0 \alpha_i &= (hmc\rho^{-1})(hmc\rho^{-1} \tilde{\beta}_i) \\
\alpha_i \alpha_j &= (hmc\rho^{-1} \tilde{\beta}_i)(hmc\rho^{-1} \tilde{\beta}_j)
\end{aligned}$$

It follows that for $i \neq 0$:

$$\begin{aligned}
\alpha_0 &= hmc\rho^{-1} \\
\alpha_i &= hmc\rho^{-1} \tilde{\beta}_i
\end{aligned}$$

Using $\beta = cdt + \tilde{\beta}$, we get:

$$\alpha = hmc\rho^{-1} \beta \tag{67}$$

Although it took a lot of work and sweat to get this neat little equality, there is a major problem. An important physical case is the case where $A = 0$. In this case, we have $\alpha = dS$ and, consequently, we get this painful sequence of equalities for $i \neq 0$:

$$\begin{aligned}
0 &= \partial_0 (hmc\rho^{-1} \tilde{\beta}_i) \\
&= \partial_0 \partial_i S \\
&= \partial_i \partial_0 S \\
&= \partial_i (hmc\rho^{-1}) \\
&= mc\rho^{-1} \partial_i h
\end{aligned}$$

Hence, h is constant. Hence, (66) implies that $m = 0$. But we do not want a vanishing mass. Hence, something went wrong. Here is a list of all the assumptions that were made to simplify the equations:

1. ρ is a constant. That's a keeper.
2. $\tilde{R}_{ij} = h^{-1}\tilde{H}_{ij}(h)$. This is a strong assumption that needs to admit a non-trivial solution. This will be checked below in §8.
3. $\tilde{\beta}$ is closed. We could discard $d\tilde{\beta} = 0$ but then we would be back to the happy carnival of notation above. Nevertheless, a trick to get rid of the above painful sequence of equalities is to use $\beta = \sum_{i=0}^{3} \beta_i dx^i$ such that $d\beta = 0$ without assuming $\beta_0 = 1$ nor that β is t-independent. This is what we will do in §7.

7. TRYING ANOTHER METRIC

Suppose again that spacetime is a Cartesian product of time and space:

$$Q = \mathbb{R} \times \tilde{Q}$$

Instead of (54), consider a metric of this kind:

$$g = h^2 \beta \otimes \beta - \tilde{g} \tag{68}$$

Here, $h : Q \to \mathbb{R}_+$ is a t-independent function. $\beta = \sum_{i=0}^{3} \beta_i dx^i \in \Omega^1(Q;\mathbb{R})$ is a closed differential 1-form which is not assumed to be t-independent and where β_0 is not assumed to equal 1. $\tilde{g}$ is a t-independent $(+,+,+)$ Riemannian metric without dt on the 3D space $\tilde{Q}$. The indices of $\tilde{g}_{ij}$ run over $i = 1, 2, 3$. More precisely:

$$\begin{aligned}
\partial_0 h &= 0 \\
d\beta &= 0 \qquad \text{(i.e. } \partial_i\beta_j = \partial_j\beta_i) \\
\tilde{g}(\partial_0, \cdot) &= \tilde{g}(\cdot, \partial_0) = 0 \\
\partial_0 \tilde{g}_{ij} &= 0
\end{aligned}$$

In terms of components:

$$g_{ij} = \begin{cases} h^2\beta_0^2 & \text{for } i = 0,\ j = 0 \\ h^2\beta_0\beta_j & \text{for } i = 0,\ j \neq 0 \\ h^2\beta_i\beta_0 & \text{for } i \neq 0,\ j = 0 \\ h^2\beta_i\beta_j - \tilde{g}_{ij} & \text{for } i \neq 0,\ j \neq 0 \end{cases}$$

The inverse matrix is:

$$g^{ij} = \begin{cases} \frac{1}{h^2\beta_0^2} - \frac{\|\beta\|_{\tilde{g}}^2}{\beta_0^2} & \text{for } i = 0,\ j = 0 \\ \frac{\beta^j}{\beta_0} & \text{for } i = 0,\ j \neq 0 \\ \frac{\beta^i}{\beta_0} & \text{for } i \neq 0,\ j = 0 \\ -\tilde{g}^{ij} & \text{for } i \neq 0,\ j \neq 0 \end{cases}$$

where for $i, j \neq 0$:

$$\begin{aligned}
\beta^i &:= \tilde{g}^{ij}\beta_j \qquad (\text{not } g^{ij}\beta_j) \\
\beta_0^2 &:= (\beta_0)^2 \\
\|\beta\|_{\tilde{g}}^2 &:= \tilde{g}^{ij}\beta_i\beta_j
\end{aligned}$$

One can verify that $g^{ik}g_{kj} = \delta^i_j$.

The new metric (68) is not a Kaluza-Klein one because β is not supposed t-independent. Because of this, we cannot use the computations done in (Williams 2015). Nevertheless, the Christoffel symbols, the Ricci components and the scalar curvature can be calculated by hand. For $i, j, k \neq 0$, let:

$$\begin{aligned}
\partial^i h &:= \tilde{g}^{ij}\partial_j h \\
\tilde{\nabla}_i\beta_j &= \partial_i\beta_j - \tilde{\Gamma}^k_{ij}\beta_k
\end{aligned}$$

The Christoffel symbols Γ^k_{ij} of g and those $\tilde{\Gamma}^k_{ij}$ of $\tilde{g}$ are related as follow. For $i, j, k, l \neq 0$:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{\partial_0\beta_0}{\beta_0} - h\beta_0\beta^l\partial_l h \\
\Gamma^0_{i0} &= \partial_i \ln(h) + \frac{\partial_i\beta_0}{\beta_0} - h\beta_i\beta^l\partial_l h \\
\Gamma^0_{0j} &= \partial_j \ln(h) + \frac{\partial_j\beta_0}{\beta_0} - h\beta_j\beta^l\partial_l h \\
\Gamma^0_{ij} &= \frac{\beta_i\partial_j h}{h\beta_0} + \frac{\beta_j\partial_i h}{h\beta_0} - \frac{\beta^l\beta_i\beta_j h\partial_l h}{\beta_0} + \frac{\tilde{\nabla}_i\beta_j}{\beta_0} \\
\Gamma^k_{00} &= h\beta_0\beta_0\partial^k h \\
\Gamma^k_{i0} &= h\beta_0\beta_i\partial^k h \\
\Gamma^k_{0j} &= h\beta_0\beta_j\partial^k h \\
\Gamma^k_{ij} &= \tilde{\Gamma}^k_{ij} + h\beta_i\beta_j\partial^k h
\end{aligned}$$

Some Mahjong skills show that the Ricci curvature tensors R_{ij} of g and $\tilde{R}_{ij}$ of $\tilde{g}$ are related for $i, j \neq 0$ as:

$$\begin{aligned}
R_{00} &= \beta_0\beta_0 h\tilde{\Delta}h \\
R_{i0} &= \beta_0\beta_i h\tilde{\Delta}h \\
R_{ij} &= \beta_i\beta_j h\tilde{\Delta}h + \tilde{R}_{ij} - h^{-1}\tilde{H}_{ij}(h)
\end{aligned}$$

It follows that the scalar curvature R of g and the scalar curvature $\tilde{R}$ of $\tilde{g}$ are related as:

$$R = -\tilde{R} + 2h^{-1}\tilde{\Delta}h \tag{69}$$

Suppose once more that for $i, j \neq 0$ we have:

$$\tilde{R}_{ij} = h^{-1}\tilde{H}_{ij}(h) \tag{70}$$

Tracing both sides of (70) by $\tilde{g}^{ij}$ gives:

$$\tilde{R} = h^{-1}\tilde{\Delta}h \tag{71}$$

The Ricci and the scalar curvature become:

$$R_{00} = \beta_0\beta_0 h\tilde{\Delta}h \tag{72}$$

$$R_{i0} = \beta_0\beta_i h\tilde{\Delta}h \tag{73}$$

$$R_{ij} = \beta_i\beta_j h\tilde{\Delta}h \tag{74}$$

$$R = \tilde{R} = h^{-1}\tilde{\Delta}h \tag{75}$$

We want to solve the EFE (36) and the hypothesis (51):

$$R_{ij} = -\frac{6}{\hbar^2}\alpha_i\alpha_j \tag{76}$$

$$R = -\frac{6}{r_C^2\rho^2} \tag{77}$$

That is, for $i, j \neq 0$ we want to solve:

$$-\frac{6}{\hbar^2}\alpha_0\alpha_0 = \beta_0\beta_0 h\tilde{\Delta}h \tag{78}$$

$$-\frac{6}{\hbar^2}\alpha_0\alpha_i = \beta_0\beta_i h\tilde{\Delta}h \tag{79}$$

$$-\frac{6}{\hbar^2}\alpha_i\alpha_j = \beta_i\beta_j h\tilde{\Delta}h \tag{80}$$

$$-\frac{6}{r_C^2\rho^2} = R = \tilde{R} = h^{-1}\tilde{\Delta}h \tag{81}$$

Doing so, for $i, j \neq 0$ we get:

$$\begin{aligned}
\alpha_0\alpha_0 &= (hmc\rho^{-1}\beta_0)(hmc\rho^{-1}\beta_0)\\
\alpha_0\alpha_i &= (hmc\rho^{-1}\beta_0)(hmc\rho^{-1}\beta_i)\\
\alpha_i\alpha_j &= (hmc\rho^{-1}\beta_i)(hmc\rho^{-1}\beta_j)
\end{aligned}$$

Hence, as in (82), we get:

$$\alpha = hmc\rho^{-1}\beta \tag{82}$$

There is no painful sequence of equalities awaiting. However, something strange is happening. A straightforward calculation using (68) and (82) shows that $v = \alpha^\sharp$ is explicitly given by:

$$v = \alpha^\sharp = g^{ij}\alpha_i\partial_j = \frac{mc}{h\rho\beta_0}\partial_0 = \frac{m^2c^2}{\rho^2\alpha_0}\partial_0 \tag{83}$$

So, the vector field $v \in \mathfrak{X}(\mathbb{R} \times \tilde{Q})$ points straight up in the time direction. This is fairly rigid and seems to prevent any interesting dynamics. On the other side, a good news is that a straightforward calculation using the Christoffel symbols of the metric (68) shows that v satisfies the divergence-free equation (39):

$$\mathrm{Div}(v) = 0$$

Recalling from (52) that the length of v is constant, a straightforward calculation shows that:

$$\begin{aligned}
\mathcal{L}_v h &= 0\\
d\alpha &= mc\rho^{-1}dh \wedge \beta = d\ln h \wedge \alpha\\
\mathcal{L}_v\alpha &= -\frac{m^2c^2}{\rho^2}d\ln h\\
\mathcal{L}_v\beta &= -\frac{mc}{\rho}\frac{dh}{h^2}\\
\mathcal{L}_v\tilde{g} &= 0\\
\mathcal{L}_v g &= -\frac{mc}{\rho}dh \odot \beta
\end{aligned}$$

In particular, v is not a Killing vector field of g.

Now, we must deal with the hypothesis (70). Is this hypothesis valid?

8. ON THE EXISTENCE OF H

The goal here is to build a metric $\tilde{g}$ that satisfies (70):

$$\tilde{R}_{ij} = h^{-1}\tilde{H}_{ij}(h) \tag{84}$$

for some function h. Inspired by the Euclidean metric on $\mathbb{R}^3$ given in spherical coordinates:

$$dr \otimes dr + r^2 \cdot (d\theta \otimes d\theta + sin^2(\theta) \cdot d\varphi \otimes d\varphi)$$

let's consider a metric of this kind:

$$\tilde{g} = f(r)^2 \cdot dr \otimes dr + r^2 \cdot (d\theta \otimes d\theta + sin^2(\theta) \cdot d\varphi \otimes d\varphi)$$

In terms of components:

$$\begin{aligned}
\tilde{g}_{11} &= f^2\\
\tilde{g}_{22} &= r^2\\
\tilde{g}_{33} &= r^2\sin^2(\theta)\\
\tilde{g}_{ij} &= 0 \qquad \text{else}
\end{aligned}$$

Its inverse matrix is given by:

$$\begin{aligned}
\tilde{g}^{11} &= 1/f^2\\
\tilde{g}^{22} &= 1/r^2\\
\tilde{g}^{33} &= 1/(r^2\sin^2(\theta))\\
\tilde{g}^{ij} &= 0 \qquad \text{else}
\end{aligned}$$

The Christoffel symbols are:

$$\begin{aligned}
\tilde{\Gamma}^1_{11} &= \partial_1 \ln f\\
\tilde{\Gamma}^1_{22} &= -rf^{-2}\\
\tilde{\Gamma}^1_{33} &= -rf^{-2} \cdot \sin^2(\theta)\\
\tilde{\Gamma}^2_{12} &= \tilde{\Gamma}^2_{21} = r^{-1}\\
\tilde{\Gamma}^2_{33} &= -\sin(\theta)\cos(\theta)\\
\tilde{\Gamma}^3_{13} &= \tilde{\Gamma}^3_{31} = r^{-1}\\
\tilde{\Gamma}^3_{23} &= \tilde{\Gamma}^3_{32} = \cot(\theta)\\
\tilde{\Gamma}^k_{ij} &= 0 \qquad \text{else}
\end{aligned}$$

The Ricci curvature is:

$$\begin{aligned}
\tilde{R}_{11} &= 2r^{-1}\partial_1 \ln f\\
\tilde{R}_{22} &= -f^{-2} + rf^{-3}\partial_1 f + 1\\
\tilde{R}_{33} &= (-f^{-2} + rf^{-3}\partial_1 f + 1)\sin^2(\theta)\\
\tilde{R}_{ij} &= 0 \qquad \text{else}
\end{aligned}$$

The scalar curvature satisfies:

$$r^2\tilde{R} = 2(1 - \partial_1(r\mu)) \tag{85}$$

where $\mu = f^{-2}$. Suppose that $h = h(r)$. It's Hessian is:

$$\begin{aligned}
\tilde{H}_{11}(h) &= \partial_1\partial_1 h - (\partial_1 \ln f)\partial_1 h\\
\tilde{H}_{22}(h) &= -\tilde{\Gamma}^1_{22}\partial_1 h = rf^{-2} \cdot \partial_1 h\\
\tilde{H}_{33}(h) &= -\tilde{\Gamma}^1_{33}\partial_1 h = rf^{-2} \cdot \sin^2(\theta) \cdot \partial_1 h\\
\tilde{H}_{ij}(h) &= 0 \qquad \text{else}
\end{aligned}$$

Its Laplacian is then:

$$\begin{aligned}\tilde{\Delta}h &= \tilde{g}^{ij}\tilde{H}_{ij}(h)\\ &= f^{-2}\cdot\left(\partial_1\partial_1 h - (\partial_1 \ln f)\,\partial_1 h + 2r^{-1}\partial_1 h\right)\end{aligned}$$

We want to find f and h that satisfy (84). This amounts to solve three equations:

$$\tilde{R}_{ii} = h^{-1}\cdot\tilde{H}_{ii}(h) \qquad i = 1, 2, 3$$

These three equations amount to solve two partial differential equations:

$$2r^{-1}\partial_1 \ln f = h^{-1}\partial_1\partial_1 h - (\partial_1 \ln f)(\partial_1 \ln h) \tag{86}$$

$$\partial_1 \ln h = \partial_1 \ln f + (f^2 - 1)r^{-1} \tag{87}$$

A first obvious, but useless, solution is given by:

$$\begin{aligned}h(r) &= 1\\ f(r) &= 1\end{aligned}$$

Another solution is the *exterior* Schwarzschild solution:

$$\begin{aligned}h(r) &= (1 - R_s/r)^{1/2}\\ f(r) &= (1 - R_s/r)^{-1/2}\end{aligned}$$

where $R_s = 2GM/c^2$ is the Schwarzschild radius of a spherical body of mass M. However, this solution is useless for us. Indeed, such a $h(r) = (1 - R_s/r)^{1/2}$ has a vanishing Laplacian $\tilde{\Delta}h$. Doing so, this solution cannot be used to define a nonvanishing mass:

$$-6\left(\frac{mc}{\hbar\rho}\right)^2 = R = \tilde{R} = h^{-1}\tilde{\Delta}h = 0$$

We need a third solution. A naive idea would be to try the *interior* Schwarzschild solution:

$$\begin{aligned}h(r) &= \frac{3}{2}(1 - R_s/R_g)^{1/2} - \frac{1}{2}(1 - r^2 R_s/R_g^3)^{1/2}\\ f(r) &= (1 - r^2 R_s/R_g^3)^{-1/2}\end{aligned}$$

where R_g is the value of r at the surface of a mass M spherical body. However, these two functions $h(r)$ and $f(r)$ are not solutions to the above two PDE's (86,87).

Let's try something else. Suppose that $\tilde{g}$ has a constant scalar curvature $\tilde{R} = -6/(r_C\rho)^2$. Using (85), we get:

$$-6r^2/(r_C\rho)^2 = 2(1 - \partial_1(r\mu))$$

where $\mu = f^{-2}$. This implies:

$$f = \mu^{-1/2} = (1 + a/r + r^2/(r_C\rho)^2)^{-1/2}$$

for some constant a. Now that we have f, we need to find h that solves (86,87). When $a = -R_s$ and $r_C = +\infty$ (or $\rho = +\infty$), this leads to the useless Schwarzschild solution. When $a = 0$, the second equation (87) leads to:

$$h = b/(1 + r^2/(r_C\rho)^2)$$

for some constant b. However, this function does not solve the first equation (86). So we need to take $a \neq 0$. But then things get nasty.

At this point it seems that the amount of assumptions we made, namely that $d\beta = 0$, that $\tilde{R}_{ij} = h^{-1}\tilde{H}_{ij}(h)$ and that $\tilde{g}$ depends only on r, is too restrictive. It might be a good idea to seek for metrics g depending also on θ and where $d\beta \neq 0$. This would lead us in so-called *Kerr solution* territories. But we would seek for a more complicated metric than merely a Kerr one because we want a constant non-vanishing spacetime scalar curvature R.

Also, as was noted back in §7, in equation (83) we found that v must point in the time direction ∂_0. Both in §6 and §7, the metric was chosen so that a time direction was singled out. Maybe it would be better to re-interpret the metric so that we single out the direction of the timelike vector field v independently of some independent background time direction.

9. CONCLUSION AND OPENING

Conclusion: The goal of this study was to reach the flat Schrödinger equation from a GR setting. Such a quest was guided by two points. First, by a curious relationship between the Einstein field equation and the WKB approximation of the KG equation. Second, by the desire to define the effective mass in geometrical terms only. Along the way, the main problem was to find a solution to the EFE. No solution was found. Doing so, the subsequent plan to tessellate space and define an averaged flat spatial metric was not even started and the FSE was not reached.

Opening: Many choices and assumptions were made along the way that could be lifted or modified to solve the EFE. As remarked in §4, adding an electromagnetic contribution $-\frac{1}{4\mu_0}\|F\|_g^2$ to the Lagrangian density (27) might be a good idea. Such a terms would not alter the effective mass of the WKB wave while it would add terms to the Ricci curvature R_{ij}. These electromagnetic added terms could match the terms of an eventually non-vanishing $d\beta$ in the EFE.

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