The Atiyah-Floer conjecture

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Outline

1. Morse theory.
2. Floer homologies.
3. Atiyah-Floer conjecture + Atiyah's arguments supporting his conjecture (+ Taubes's bonus).
4. Various approaches and attempts($\Longleftrightarrow$ $T < 20$min.).

1.1 - Morse theory (20's)

Morse :

Get topological data of a manifold M by considering the critical set

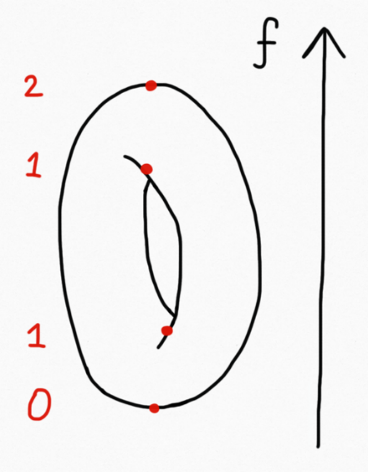
$$\mathrm{crit}(f) := \{x \in M : \mathrm{d}f|_x = 0\}$$

of a Morse function

$$f : M \to \mathbb{R}$$

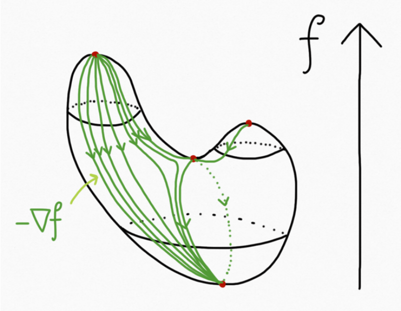
$\rightsquigarrow \chi(M) = \sum_{i=0}^{n} (-1)^i c_i(f).$

$$\left(\chi(M) := \sum_{i=0}^{n} (-1)^i b_i(M) \right)$$



*(*try it now*)*

1.2 - Morse homology (40's $\rightsquigarrow$ 60's)



Thom, Smale & Milnor :

Morse-Smale pair (f, g).

$\rightsquigarrow$ Complex generated by crit(f).

$\rightsquigarrow$ ∂ couts gradient curves.

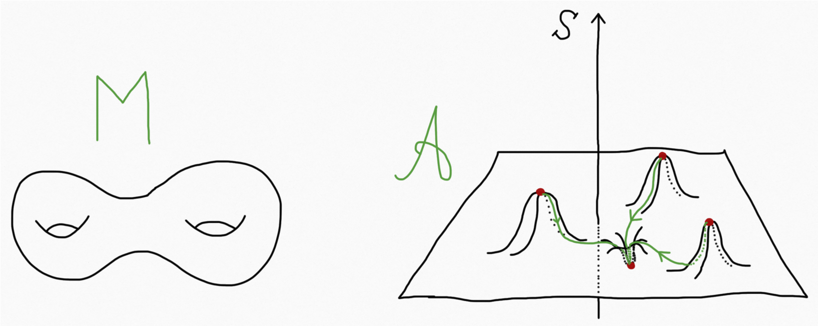
$\rightsquigarrow$ $\partial^2 = 0$

$\rightsquigarrow$ Morse homology !

$$H_*(M; \mathbb{Z}) \cong H_*(M, (f, g))$$

2.1 - Floer homologies (80's)

Floer [2, 3] : Studying the topology of a manifold M by counting "gradient curves" joining critical points of some $S : \mathcal{A} \to \mathbb{R}$ on an auxiliary space $\mathcal{A}$.

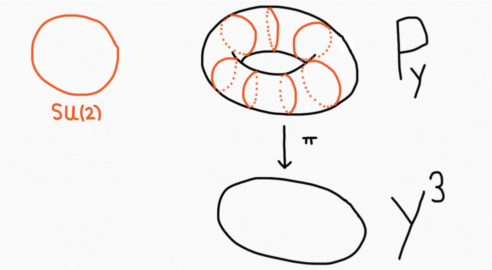


$\rightsquigarrow$ Floer homology $\mathrm{HF}_*(M) \neq H_*(M)$ ($\Rightarrow$ new invariants).

2.2 - Instanton Floer homology (80's)

Context : (necessarily) trivial SU(2)-principal bundle

$\mathrm{SU}(2) \hookrightarrow P_Y \xrightarrow{\pi} Y^3$ where Y^3 is $\mathbb{Z}\mathrm{H}S^3$, i.e. $H_*(M;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$



Auxiliary space : the space of connexions on P_Y :

$$\mathcal{A}_Y \cong \Omega^1(Y;\mathfrak{su}(2)) \quad \text{(+ technical properties)}$$

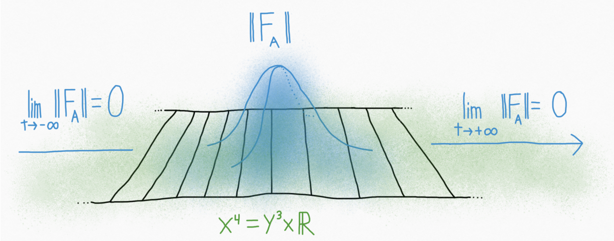
2.3 - Instanton Floer homology

Auxiliary S : Chern-Simons functional

$$S_{\mathrm{CS}} : \mathcal{A}_Y \to \mathbb{R} \,; \quad S_{\mathrm{CS}}(A) = \int_Y \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right)$$

Here, $\mathrm{crit}(S_{\mathrm{CS}}) = \mathcal{A}_Y^{\mathrm{fl}}$ and "gradient curves" describe a connexion over $X^4 = Y^3 \times \mathbb{R}$ with ASD curvature form F_A :

$$\star_g F_A = -F_A$$



2.4 - Lagr. int. Floer homology (80's-...)

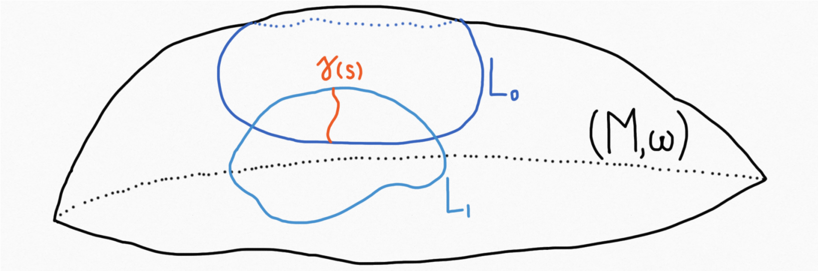
Context : Symplectic manifold (M^{2n}, ω)

$$\left(\omega \in \Omega^2(M) \quad \& \quad \mathrm{d}\omega = 0 \quad \& \quad \omega \text{ non-degen.}\right)$$

endowed with two Lagrangian submanifolds $L_0, L_1 \hookrightarrow M$

$$\left(\dim L_0 = \dim L_1 = n \quad \& \quad \omega|_{L_0} = \omega|_{L_1} = 0\right)$$

Auxiliary space : $\Gamma := \{\gamma : [0,1] \to M | \gamma(0) \in L_0, \gamma(1) \in L_1\}$



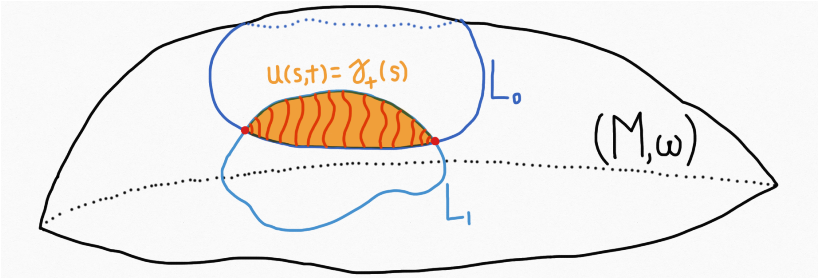
2.5 - Lagr. int. Floer homology

Auxiliary S : Consider $S : \Gamma \to \mathbb{R}$ such that :

$$(\mathrm{d}S)|_\gamma(\xi) = \int_0^1 \omega(\dot\gamma(t), \xi(t))\mathrm{d}t, \quad \forall \gamma \in \Gamma, \forall \xi \in T_\gamma\Gamma = \Gamma^\infty(\gamma^* TM)$$

Here, $\mathrm{crit}(S) \cong L_0 \cap L_1$ and "gradient curves of S" describes J-holomorphic curves in M :

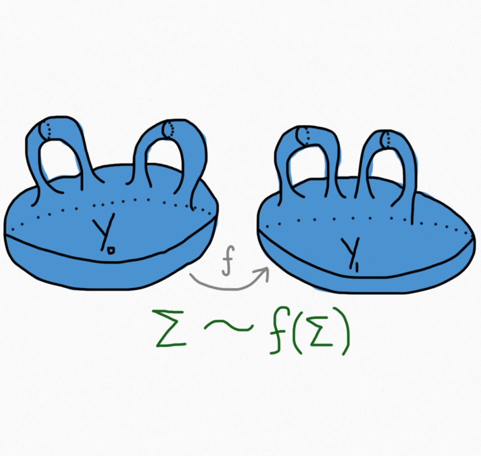
$$u : [0,1] \times \mathbb{R} \to M \quad \text{where} \quad \overline{\partial} u := \partial_s u + J\partial_t u = 0$$

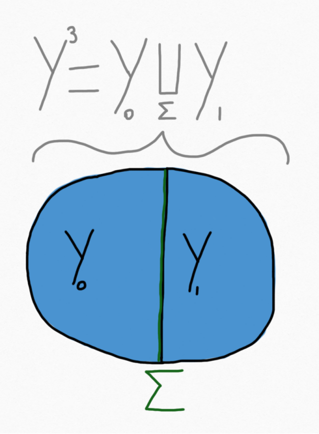


$\rightsquigarrow \mathrm{HF}_*^{\mathrm{lag}}(M, L_0, L_1)$ (*certain conditions apply*).

3.1 - Atiyah-Floer conjecture (80's)

<u>*Context of the conjecture :*</u> Let $Y = Y_0 \underset{\Sigma \sim f(\Sigma)}{\sqcup} Y_1$ be a $\mathbb{Z}HS^3$ endowed with a Heegaard splitting (here $\Sigma = \partial Y_0$ and $f(\Sigma) = \partial Y_1$ where Y_0, Y_1 are handlebodies and where $f \in \mathrm{MCG}(\Sigma)$). Visually :

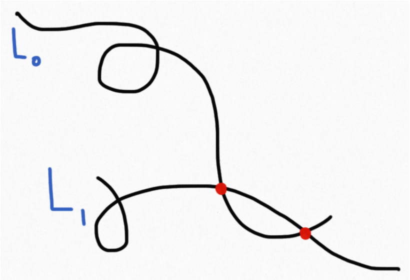




3.2 - Atiyah-Floer conjecture

The set of (gauge classes of) flat connexions over Σ that extends to (gauge classes of) flat connexions over the handlebody Y_0 (resp. Y_1) describes a singular and immerged (after perturbations) Lagrangian submanifold L_0 (resp. L_1) in the symplectic orbifold "moduli space of flat connexions over Σ" :

$$L_0, L_1 \subset \mathcal{M}_\Sigma^{\text{fl}} := \mathcal{A}_\Sigma^{\text{fl}} / \mathcal{G}_\Sigma$$



3.3 - Atiyah-Floer conjecture

Atiyah [1] : $\mathrm{HF}_*^{\mathrm{lag}}(\mathcal{M}_\Sigma^{\mathrm{fl}}, L_0, L_1) \cong \mathrm{HF}_*^{\mathrm{inst}}(Y)\,?$

3.4 - Atiyah's 1st argument

Atiyah's 1st argument :

The bijection

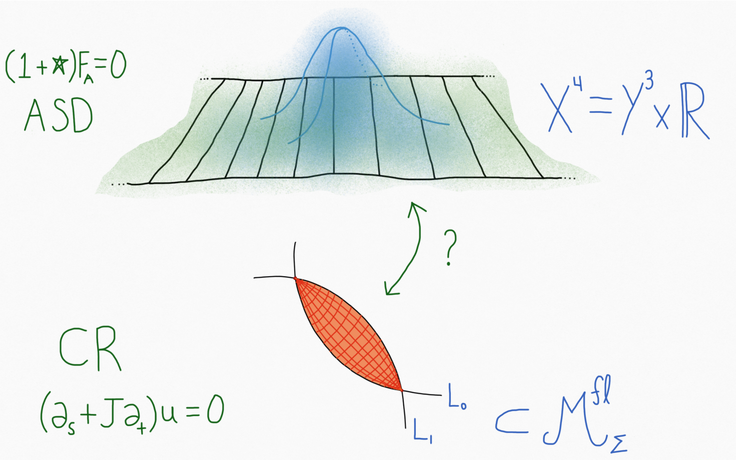
$$L_0 \cap L_1 \leftrightarrow \mathcal{M}_Y^{\text{fl}}$$

matches both Floer's complex's generators

$$\text{CF}_*^{\text{lag}}(\mathcal{M}_\Sigma^{\text{fl}}, L_0, L_1) \quad \text{and} \quad \text{CF}_*^{\text{inst}}(Y)$$

3.5 - Atiyah's 1st argument

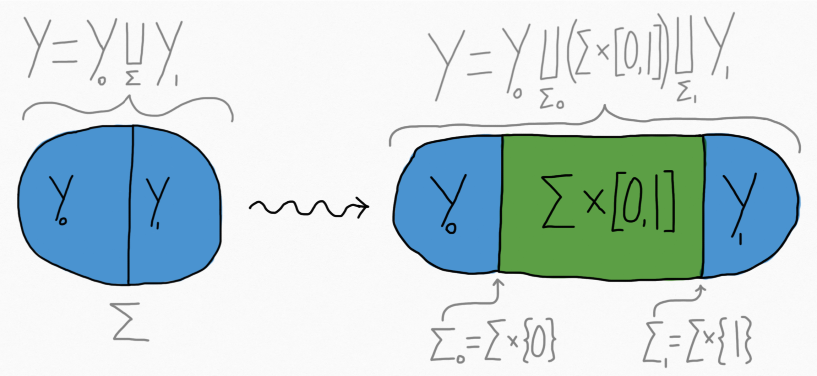
⇝ It remains to show that both complex's differential matches.
i.e. :



3.6 - Atiyah's 2nd argument

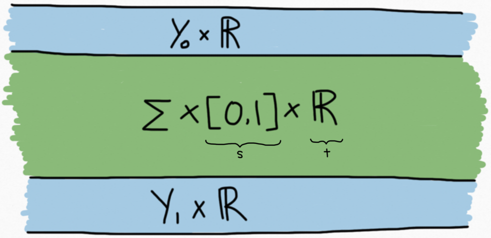
Atiyah's 2nd argument (ASD⇝CR) : Thicken Σ to $\Sigma \times [0,1]$:

$$Y = Y_0 \underset{\Sigma_0}{\sqcup} (\Sigma \times [0,1]) \underset{\Sigma_1}{\sqcup} Y_1$$



3.7 - Atiyah's 2nd argument

We are interested in the $\Sigma^2 \times [0,1] \times \mathbb{R}$ part of $X^4 = Y^3 \times \mathbb{R}$:



In temporal gauge, the ASD equation $\star_g F_A = -F_A$ decomposes, for a family of (Riem.) metrics $g_\lambda = \lambda^{-1} g_\Sigma + \lambda^2 \mathrm{d}s^2 + \mathrm{d}t^2$, $\lambda \in \mathbb{R}_{>0}$, in two equations :

$$(1) \qquad F_{A_{s,t}} = \lambda^{-1} \star_{g_\Sigma} \partial_t \varphi_{s,t}$$

$$(2) \qquad \lambda^{-1} \partial_s A_{s,t} + \star_{g_\Sigma} \partial_t A_{s,t} = \mathrm{d}_{A_{s,t}} \varphi_{s,t}$$

3.8 - Atiyah's 2nd argument

Consider this surface :

$$u(s,t) := A_{s,t} : [0,1] \times \mathbb{R} \to \mathcal{A}_\Sigma$$

and this complex structure over $\mathcal{A}_\Sigma$:

$$J := \star_{g_\Sigma} \quad \text{(because } \star_{g_\Sigma}^2 = -1 \text{ over } \Omega^1(\Sigma; \mathfrak{su}(2)) \text{)}$$

$\Rightarrow$ equation (2) reformulates as

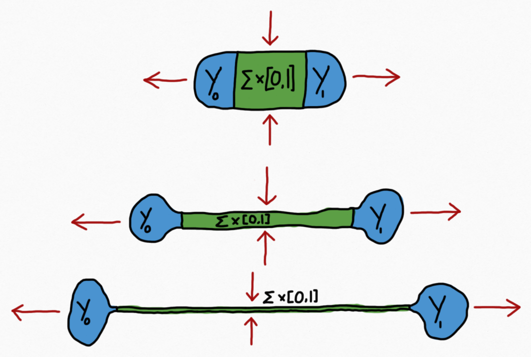
$$(3) \qquad \lambda^{-1} \partial_s u(s,t) + J \partial_t u(s,t) = \mathrm{d}_{A_{s,t}} \varphi_{s,t}$$

$\rightsquigarrow$ that's *almost* the CR eq. of J-holom. curves

$$\overline{\partial} u = 0$$

3.9 - Atiyah's 2nd argument

To have u resting in $\mathcal{A}_{\Sigma}^{\text{fl}}$, Atiyah proceeds by adiabatic limit by *stretching the neck* $\Sigma \times [0, 1]$ of Y :



Indeed : $\lim_{\lambda \to +\infty} F_{A_{s,t}} = \lim_{\lambda \to +\infty} \lambda^{-1} \star_{g_\Sigma} \varphi_{s,t}^{(t)} = 0.$

3.10 - Atiyah's 2nd argument

Problem 1 : $\lim_{\lambda\to+\infty}(3)$ is $J\partial_t u(s,t) = \mathrm{d}_{A_{s,t}}\varphi_{s,t}$.

Which is *less* CR than before...

Problem 2 : are Lagrangian boundary conditions (of u in $(\mathcal{M}_\Sigma^{\mathrm{fl}}, L_0, L_1)$) verified ?

Problem 3 : Is Atiyah's *ASD⇝CR procedure* a bijection ?

Nevertheless, bonus argument in favor of the conjecture :
Taubes [6] showed that both complex's Euler characteristic matches.

4.1 - Various approaches and attempts

Atiyah's original conjecture is still open.

Salamon and Wehrheim's approach is about instantons with lagrangian boundary conditions.

There are various variants of the conjecture :

- *Floer's variant :* non-trivial SO(3)-bundle over mapping torus. Proved by Dostoglou & Salamon.
- *Fukaya's variant* : non-trivial SO(3)-bundle & hybrid ASD/CR equation
- and many other versions/attempts proposed by *Wehrheim*, *Woodward*, *Manolescu*, *Duncan*, *Lipyanskiy*, *Yoshida*, *Lee*, *Li*, etc.

[1] M. F. Atiyah, *New invariants of 3- and 4-dimensional manifolds*, Proceedings of Symposia in Pure Mathematics **48** (1988).

[2] A. Floer, *An instanton-invariant for 3-manifolds*, Commun. Math. Phys. **118** (1988), 215–240.

[3] ______, *Morse theory for lagrangian intersections*, J. Differential Geometry **28** (1988), no. 3, 513–547.

[4] K. Fukaya, *Morse homotopy, A^{∞}-category, and floer homologies*, Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993), Lecture Notes Series, vol. 18, Seoul National University, 1993, pp. 1–102.

[5] D. A. Salamon, *Lagrangian intersections, 3-manifolds with boundary, and the atiyah-floer conjecture*, Proceedings of the International Congress of Mathematics, Birkäuser Verlag (1994).

[6] C. H. Taubes, *Casson's invariant and gauge theory*, J. Differential Geometry **31** (1990), 547–599.